SEMICLASSICAL ANALYSIS OF MAGNETIC WEYL OPERATORS ON A TORUS PHASE SPACE

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Sometimes a scream is better than a thesis.

(Ralph Waldo Emerson)

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Abstract

In this thesis the semiclassical trace formula and the spectra of the magnetic Laplacian on a torus phase space are studied. Developing the non-magnetic and magnetic Weyl calculus on $\mathbb{R}^{2n}$ an association between continuous functions and operators on some complex Hilbert can be obtained. By restricting the phase space to a torus, $\mathbb{T}^{2n}$, the quantisation takes the form of a series expansion of discrete translation operators. Considering the translations around the torus, phase space and magnetic flux quantisations are obtained. With an expression for the discrete Laplacian the classical Hamiltonian is obtained which when quantised yields the Laplacian. Investigating the infinite momentum limit, a relationship between the corresponding quantum system on the configuration space torus is found to be equivalent in this limit. Representing the translation operators as finite dimensional matrices, eigenvalues are obtained numerically, and spectral statistics are computed and compared with that from random matrix theory. Finally, semiclassical trace formulae are studied for the free particle and for the Landau gauge.

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Chapter 1

Introduction

Over the previous century quantum mechanics has been tested and verified yielding one of our most accurate theories to describe the dynamics of microscopic particles. At these small scales the intuitive theory of classical mechanics breaks down. Experiments showed that certain quantities were being quantised. The energy of an atom with respect to time which at present, sits in its first excited state, then spontaneously jumps down to its ground state cannot be described by the classical theory; it was unable to account for the discontinuity in the energy levels. These paradoxical predictions flew in the face of classical mechanics. For this reason many pioneers including Einstein, tried to hold on to the theory of classical mechanics, ridiculing the new quantum theory by saying, "God does not play dice" suggesting the randomness and paradoxical arguments were just the frontman of some deeper vein of truth.

Discrete energy levels reared its head with the introduction of a constant magnetic field. In two dimensions, the classical picture showed a continuous energy range whereas the quantum analogue exhibited discrete energy levels in the form of Landau levels. These energy levels are found to be functions of Planck's constant, $\hbar$ and an integer, n which labels the level and is seen to be elements of a discrete set, $\{E_0(\hbar), E_1(\hbar), \cdots\}$. One may propose that the limit when these discontinuities becomes infinitely small be defined as the semiclassical limit; mathematically this is the limit where $\hbar \to 0$ and is the focus of semiclassical mechanics.

Semiclassical mechanics has two main avenues of study:

1. Given classical observables, how does one quantise them? For example, given a Hamiltonian, H on phase space, N how does one acquire the mapping $H \mapsto \mathrm{Op}[H]$ and what properties does it have to possess in order to agree with observed experiments?

2. How does a dynamical system, obeying the laws of classical mechanics, determine the behaviour of the Schrödinger equation as $\hbar \to 0$,

$$\mathrm{Op}[H]\psi = E\psi? \tag{1.0.1}$$

   Or, what can the dynamics of a classical system tell us about certain spectral functions as $\hbar \to 0$?

The first use of semiclassical methods was developed around 1920 by using the Bohr-Sommerfeld model (aimed at avenue 2), where the action around a closed trajectory is used to find the spectrum of a bound system [Mes99]. Later, Einstein [Ein17], extended this to non-separable systems but requiring the classical system to have as many conserved quantities as degrees-of-freedom. When this condition is invalid the model does not hold and he hinted towards a solution in the form of the ergodic principle. The development of caustics in the classical motion, added correction terms to the Bohr-Sommerfeld model which culminated in the Einstein-Brillouin-Keller (EBK) method. The quantum spectrum obtained from integrable systems was coined 'regular', [Per73], as the classical motion was confined to well defined invariant surfaces in phase space. It was becoming clear that a relation between the classical dynamics and the quantum spectrum was emerging.

One milestone of semiclassical mechanics was the development of the trace formula, which relates quantum spectral functions to the periodic orbits on phase space. The first formula, derived by Gutzwiller, [Gut70, Gut71] focuses on the density of states and is used for chaotic systems where the periodic orbits are isolated. This formula breaks down however, when integrable systems are studied since the periodic orbits now occur in continuous families - this was soon overcome by the Berry and Tabor trace formula [MVB76, MVB77a]. Generalisations to the trace formula occurred years later dealing with continuous symmetries [SCC91], which re-derived the Berry-Tabor trace formula and extended the validity to non-Abelian symmetries [SCC92].

The main focus has so far been aimed at avenue 2. In order to study quantum mechanics in the semiclassical limit, self-adjoint operators which represent the observable quantum mechanically (avenue 1) have to be obtained. This process from which a self-adjoint operator is obtained from the knowledge of a classical observable is named quantisation. Weyl [Wey27], developed a theory of quantisation which assigns a continuous function on phase space to a linear self-adjoint operator on an infinite dimensional Hilbert space typically L^2, over some configuration space. The operators studied in this thesis are obtained by quantising classical observables on a torus phase space, $\mathbb{T}^{2n}$. Restricting the phase space to torus $\mathbb{T}^{2n}$, results in observables which are periodic in both n, position and momentum variables, resulting in states which belong to a finite dimensional Hilbert space with dimension N^n. All N^n dimensional complex Hilbert spaces are isomorphic to $\mathbb{C}^{N^n}$ and thus quantum observables are finite dimensional Hermitian matrices and can be studied by methods of Linear Algebra.

The focus of this thesis will be the Weyl quantisation of the Laplacian in the presence of a magnetic field on a torus phase space. We will study the distribution of its eigenvalues with homogeneous and non-homogeneous magnetic fields as well as the of the corresponding trace formulae.

The structure of the thesis is as follows. In chapter 2 we develop the basics of Hamiltonian mechanics and discuss the manifold structure of mechanics; although we will not explicitly use the manifold structure, it does provide the mathematical structure of phase space which is the arena in which Hamiltonian mechanics is studied. Having studied what phase space is, we develop the notion of a toroidal phase space from which the classical observables will be defined. We discuss the importance of constants of motion and discuss how they simplify the motion of the dynamical system by restricting motion to level surfaces. From here we develop the notion of a canonical transformation which allows one great freedom to pick local coordinates on phase space, reducing the complexity of certain problems. We will take advantage of the constants of motion and the restriction they impose for the trajectories and use this to obtain the well known Hamilton-Jacobi equation (HJE). Restricting the level surfaces to be compact, we develop a coordinate system which reduces the equations of motion which can be solved by quadratures. A system confined to compact levels surfaces which in number, are equal to the number of degree-of-freedom in phase space, is said to be Liouville integrable. After discussing the implications this has on the motion of the system we end with an example of how the HJE is used to obtain the complete solution for a charged particle in a constant magnetic field.

Chapter 3 develops the theory of non-magnetic and magnetic Weyl calculus on $\mathbb{R}^{2n}$ and $\mathbb{T}^{2n}$. We first start with some basic theorems from linear algebra and the theory of finite-dimensional Hilbert spaces applied to quantum mechanics. Weyl quantisation is then obtained via the irreducible representation of the Heisenberg group. We express the quantisation as an operator expansion in terms of a symplectic Fourier transform and phase space translation operators, $U(\boldsymbol{q}, \boldsymbol{p})$:

$$\mathrm{Op}_\hbar^W[f] = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{q}\, \mathrm{d}\boldsymbol{p}\, (\mathcal{F}_\hbar f)\, (\boldsymbol{q}, \boldsymbol{p}) U(\boldsymbol{p}, \boldsymbol{q}), \tag{1.0.2}$$

where

$$U(\boldsymbol{p}, \boldsymbol{q}) = \mathrm{e}^{\frac{i}{\hbar}\boldsymbol{p}\cdot\hat{\boldsymbol{Q}} - \frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\boldsymbol{P}}} \tag{1.0.3}$$

and $\boldsymbol{p} \cdot \hat{\boldsymbol{Q}} = p_1\hat{Q}^1 + \cdots + p_n\hat{Q}^n$ such that $\hat{Q}^i$ is a multiplication operator $\hat{Q}^i\psi(\boldsymbol{x}) = x^i\psi(\boldsymbol{x})$ and $\boldsymbol{q} \cdot \hat{\boldsymbol{P}} = q^1\hat{P}_1 + \cdots + q^n\hat{P}_n$ is defined such that, $\hat{P}_i := -i\hbar\partial_{x^i}$ where $\partial_{x^i} = \partial/\partial x^i$. Having developed a general scheme for quantising continuous functions on $\mathbb{R}^{2n}$ we restrict the phase space to that of a torus, $\mathbb{T}^{2n}$. The restriction of periodicity on the phase space translation operators imposes a condition that the area of phase space, $\mathbb{T}^2$ be quantised:

$$\ell_q\ell_p = 2\pi\hbar N, \tag{1.0.4}$$

where N is the dimension of the Hilbert space and ℓ_q and ℓ_p are the length parameters of the phase space. This result was shown in [JH80, NB89, Esp93, AB96, ME14] for phase space $\mathbb{T}^2$. Next, we obtain an association with the states in $L^2(\mathbb{T}^n)$ with the vectors in $\mathbb{C}^{N^n}$ and reduce the theory of Weyl quantisation on the torus to that of finite dimensional matrices. From (1.0.4) the dimension of the Hilbert space tends to infinity in the semiclassical limit, thus the semiclassical limit is equivalent to the limit of large matrices. We next obtain the quantisation of a periodic symbol by replacing the position and momentum coordinates in

$$f(\boldsymbol{q}, \boldsymbol{p}) = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}} f_{\boldsymbol{n},\boldsymbol{m}} \mathrm{e}^{2\pi i\left(\frac{\langle \boldsymbol{n},\boldsymbol{q}\rangle}{\ell_q} - \frac{\langle \boldsymbol{m},\boldsymbol{p}\rangle}{\ell_p}\right)}, \tag{1.0.5}$$

with their corresponding operators to obtain an expansion in terms of discrete translation operators

$$\mathrm{Op}_\hbar^W[f] = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}} f_{\boldsymbol{n},\boldsymbol{m}} T^{\boldsymbol{m},\boldsymbol{n}}. \tag{1.0.6}$$

We then discuss the generalisation of Weyl quantisation to include a magnetic field on $\mathbb{R}^{2n}$. To obtain a gauge covariant theory, rather than using the minimal-coupling principle in the non-magnetic Weyl calculus we develop the quantisation based on the magnetic translation operators:

$$U^{\boldsymbol{A}}(\boldsymbol{p},\boldsymbol{q}) = \mathrm{e}^{\frac{i}{\hbar}\left(\boldsymbol{p}\cdot\hat{\boldsymbol{Q}}-\boldsymbol{q}\cdot\hat{\boldsymbol{P}}+\boldsymbol{q}\cdot\boldsymbol{A}(\hat{\boldsymbol{Q}})\right)}, \tag{1.0.7}$$

The restriction to periodic symbols imposes a flux quantisation condition similar to that of the Dirac monopole [Dir31]

$$b\ell_q^2 = 2\pi\hbar M, \tag{1.0.8}$$

where b is the magnetic field strength and ℓ_q is the length parameter of the torus configuration manifold. It is shown that in the presence of a homogeneous, and non-homogeneous magnetic field, a flux quantisation condition is obtained; this is a natural implication of the periodicity of the torus. We finally obtain the quantisation of a periodic symbol in the presence of a magnetic field as a Fourier expansion of magnetic translation operators similar to that for the non-magnetic case

$$\mathrm{Op}_\hbar^{\mathcal{A}}[f] = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}} f_{\boldsymbol{n},\boldsymbol{m}} T_{\mathcal{A}}^{\boldsymbol{m},\boldsymbol{n}}. \tag{1.0.9}$$

In chapter 4 we introduce the Laplacian operator in terms of non-magnetic and magnetic translation operators. We prove a result that allows us to obtain a classical Hamiltonian for the Landau gauge via minimal-coupling only when $M/N \in \mathbb{Z}$. We discuss the infinite volume limit and show that this is equivalent to a charged particle on the torus configuration space in the presence of a magnetic field. We also discuss the theory of EBK-quantisation for the non-magnetic Laplacian and develop an algorithm to obtain the Weyl quantisation of a Hamiltonian operator for general magnetic fields. It is found that the eigenvalues of the Hamiltonian matrix agrees well with the eigenvalues obtained via EBK-quantisation. The eigenvalues are then studied and spectral statistics are used to describe the nature of the spectra, either regular or chaotic by comparing the relative spacing distribution to that of random matrix ensembles.

In chapter 5 we discuss the theory behind the trace formula for general, as well as integrable systems. From the trace formula, one can pick a suitable test function and obtain information about the classical periodic orbits from the spectra of the Hamiltonian operator. We then use the Berry-Tabor trace formula [MVB76, MVB77a], to calculate the trace formula for the non-magnetic case as well as in the Landau gauge and use a suitable test function to obtain information about periodic orbits for the case of the homogenous and non-homogeneous magnetic field.

Chapter 2

Classical Mechanics

In this section we introduce some definitions and techniques used in the thesis which will form the basis of the work on classical dynamics. The material of this chapter follows loosely that of [JJS98,KVA97,Lan76].

2.1 Manifold structure of mechanics

A particle's position is defined locally in terms of coordinates $\boldsymbol{q} = (q^1, \cdots, q^n)$ on a manifold M, called the configuration space manifold. To specify the motion on the configuration space, the velocity $\dot{\boldsymbol{q}} = (\dot{q}^1, \dot{q}^2, \cdots, \dot{q}^n)$, where the dot indicates differentiation with respect to time, at $\boldsymbol{x} \in M$ defines the tangent space, $T_{\boldsymbol{x}}M$. The Tangent bundle, TM is defined as the union of the tangent space $T_{\boldsymbol{x}}M$ at all points in M and has a local coordinate representation of $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ (see [KVA97] for an in-depth proof of the manifold structure of the tangent bundle). The state of a classical system is represented as a function, $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) : TM \times \mathbb{R} \to \mathbb{R}$. The equations of motion are the solutions to the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^j}\right) - \frac{\partial L}{\partial q^j} = 0 \text{ for } j = 1, \cdots n. \tag{2.1.1}$$

The dynamical systems we will be studying in this thesis are those which evolve in phase space. Phase space locally has coordinates of position and momentum and thus Lagrangian mechanics requires a modification in the form of a Legendre transformation which leads to Hamiltonian mechanics. Before studying the implications of the Legendre transformation on the Lagrangian, we first define the torus. Defining a lattice by

$$\Gamma = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2 \oplus \cdots \mathbb{Z}L_{2n} := \left\{ \sum_{i=1}^{2n} L_i s_i ; s_i \in \mathbb{Z}, i = 1, 2, \cdots, 2n \right\}, \tag{2.1.2}$$

the $2n$-torus is the set

$$\mathbb{T}^{2n} := \mathbb{R}^{2n}/\Gamma, \tag{2.1.3}$$

which is the collection of equivalence classes on $\mathbb{R}^{2n}$ where "$\sim$" is defined as $T_1 \sim T_2$ if

$$T_1 - T_2 \in \Gamma. \tag{2.1.4}$$

This equivalence class assigns periodic boundary conditions to $\mathbb{R}^{2n}$ with periods $L_1, L_2, \cdots, L_{2n}$. Functions on $\mathbb{T}^{2n}$ can therefore be expanded in a Fourier series with respect to the periods $L_1, L_2, \cdots, L_{2n}$ and thus can be considered to be $\mathbb{R}^{2n}$ with periodic boundary conditions. Therefore locally expressions which hold in $\mathbb{R}^{2n}$ hold in $\mathbb{T}^{2n}$. In what follows we define the torus to be

$$\mathbb{T}^{2n} = \mathbb{R}^{2n}/\left(\underbrace{\ell_q\mathbb{Z} \oplus \cdots \oplus \ell_q\mathbb{Z}}_{n} \oplus \underbrace{\ell_p\mathbb{Z} \oplus \cdots \oplus \ell_p\mathbb{Z}}_{n} \right), \tag{2.1.5}$$

where $L_i = \ell_q$ for $i = 1, 2, \cdots n$ and $L_i = \ell_p$ for $i = n+1, n+2, \cdots, 2n$. In this thesis when we mention a function f on phase space, we mean $f \in C^\infty(\mathbb{T}^{2n})$, or, an infinitely differentiable continuous function, f over $\mathbb{T}^{2n}$.

We have seen that Lagrangian mechanics studies vector fields on the tangent bundle. Hamiltonian mechanics is the study of vector fields on the cotangent bundle.

A cotangent vector (or in the language of differential forms, a one-form) is a linear function which takes elements of $T_{\boldsymbol{q}}\mathbb{R}^n$ to functions on $\mathbb{R}^n$ and is an element of the dual space to $T_{\boldsymbol{q}}\mathbb{R}^n$. This dual space is called the cotangent space at $\boldsymbol{q}$ and is denoted as $T^*_{\boldsymbol{q}}\mathbb{R}^n$. The basis of the cotangent space is denoted as $\{\mathrm{d}q^i\}$ where a local coordinate representation of a cotangent vector is written as

$$\omega^1 = \sum_{i=1}^{n} p_i \mathrm{d}q^i. \tag{2.1.6}$$

The duality is seen by

$$\mathrm{d}q^i(\hat{e}_j) = \delta^i_j \tag{2.1.7}$$

where $\hat{e}_j$ is the basis of $T\mathbb{R}^n$. Thus, applying a vector $\boldsymbol{V} = V^j\hat{e}_j \in T\mathbb{R}^n$ to ω^1 yields

$$\omega^1(\boldsymbol{V}) = \omega^1\left(\sum_{j=1}^{n} V^j\hat{e}_j\right) = \sum_{i,j=1}^{n} p_i V^j \mathrm{d}q^i(\hat{e}_j) = \sum_{i,j=1}^{n} p_i V^j \delta^i_j = \sum_{i=1}^{n} p_i V^i. \tag{2.1.8}$$

The cotangent bundle, $T^*\mathbb{R}^n := \cup_{\boldsymbol{q}\in\mathbb{R}^n} T^*_{\boldsymbol{q}}\mathbb{R}^n$, is the union of the cotangent spaces at all points in $\mathbb{R}^n$. Normally one would need to take care as transition functions are needed between coordinate patches. For our purposes we will not be concerned about this and specify the basis of the cotangent bundle as $\{\mathrm{d}q^i\}$. An element of $T^*\mathbb{R}^n$ is a one-form on the tangent space to $\mathbb{R}^n$ at some point $\boldsymbol{q} \in \mathbb{R}^n$. The local coordinate representation is the n points of $\boldsymbol{q}$ and the n-components $\boldsymbol{p} = (p_1, \cdots, p_n)$ of the one-form, ω^1. The $2n$ components $(\boldsymbol{q}, \boldsymbol{p})$, make up the local coordinates of the cotangent bundle.

The cotangent bundle has a natural symplectic structure given by the differential two-form which is found by applying the exterior derivative to (2.1.6 i.e.,

$$\omega^2 = \mathrm{d}\omega^1 = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q^i \tag{2.1.9}$$

where

$$(\omega_1^1 \wedge \omega_2^1)(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = \omega_1^1(\boldsymbol{\eta}_1)\omega_2^1(\boldsymbol{\eta}_2) - \omega_2^1(\boldsymbol{\eta}_1)\omega_1^1(\boldsymbol{\eta}_2)$$

for two one-forms ω_1^1 and ω_2^1 and two vectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ in $\mathbb{R}^n$. We let $\boldsymbol{z} = (\boldsymbol{y}, \boldsymbol{l})$ and $\boldsymbol{w} = (\boldsymbol{q}, \boldsymbol{p})$ be the components of two vector fields on $\mathbb{R}^{2n}$. The symplectic form is seen to be the area element

$$\omega^2(\boldsymbol{w}, \boldsymbol{z}) := \sigma(\boldsymbol{w}, \boldsymbol{z}) = \langle \boldsymbol{y}, \boldsymbol{p} \rangle_{\mathbb{R}^{2n}} - \langle \boldsymbol{q}, \boldsymbol{l} \rangle_{\mathbb{R}^{2n}}, \tag{2.1.10}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^{2n}}$ is the inner product on $\mathbb{R}^{2n}$.

A symplectic manifold is the pairing (M, ω^2) where M is a differentiable manifold and ω^2, a closed ($\mathrm{d}\omega^2 = 0$), non-degenerate (if $\omega^2(\boldsymbol{\eta}, \boldsymbol{\xi}) = 0$ for all $\boldsymbol{\xi}$, then $\boldsymbol{\eta} = 0$) two-form. We will refer to the cotangent bundle with the symplectic two-form $\omega^2 = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q^i$ as the phase space of the dynamical system which has dimension $2n$.

We now replace $T^*\mathbb{R}^n$ with $\mathbb{T}^{2n}$ and refer to $(\mathbb{T}^{2n}, \omega := \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q^i)$ as the phase space, N.

The equations of motion for a system with Hamiltonian $H(\boldsymbol{q}, \boldsymbol{p})$ (the energy function) on a phase space N, is defined as the solution to the following differential equations

$$\dot{p}_i = \frac{\partial H}{\partial q^i}, \tag{2.1.11a}$$

$$\dot{q}^i = -\frac{\partial H}{\partial p_i}. \tag{2.1.11b}$$

We say that the motion on N is the flow of a Hamiltonian vector field generated by H. Hamilton's equations can also be defined using the Poisson bracket. The Poisson bracket between two functions f and g is defined as

$$\{f, g\}_{qp} = \sum_{i=1}^n \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}. \tag{2.1.12}$$

Hamilton's equations can thus be expressed as

$$\dot{q}^i = \{q^i, H\}_{qp}, \tag{2.1.13}$$

$$\dot{p}_i = \{p_i, H\}_{qp}. \tag{2.1.14}$$

From this we can also define the time derivative of a function f as

$$\dot{f} = \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\mathrm{d}q^i}{\mathrm{d}t} + \frac{\partial f}{\partial p_i} \frac{\mathrm{d}p_i}{\mathrm{d}t} \right) \tag{2.1.15}$$

$$= \frac{\partial f}{\partial t} + \{f, H\}_{qp}, \tag{2.1.16}$$

where we used (2.1.11a), (2.1.11b) and (2.1.12). We see that if the function f is time-independent, then to be a constant of motion ($\dot{f} = 0$) the function's Poisson bracket with the Hamiltonian is zero. Therefore, $H(\boldsymbol{q}, \boldsymbol{p}, t)$ the total energy, is a constant of motion if it is time independent. Throughout this thesis we will be interested in Hamiltonians which are time independent and thus the energy is conserved. We will therefore always have one constant of motion, E.

Finding constants of motion simplifies the dynamics since it reduces the available phase space in which the system evolves. This can be illustrated by the following example. Given a Hamiltonian,

$$H(q^2, p_1, p_2) = E, \tag{2.1.17}$$

and phase space $(\mathbb{R}^4, \omega^2 = \sum_{i=1}^{2} \mathrm{d}p_i \wedge \mathrm{d}q^i)$, we know from (2.1.11a) and (2.1.11b) that

$$\dot{q}^1 = \frac{\partial H}{\partial p_1}, \; \dot{q}^2 = \frac{\partial H}{\partial p_2}, \; \dot{p}_2 = -\frac{\partial H}{\partial q^2}, \text{ and } \dot{p}_1 = 0. \tag{2.1.18}$$

Due to the existence of a constant of motion, the dynamics have become that of a system of two equations, eliminating one variable with $\dot{p}_1 = 0$ and another with $H(q^2, p_1, p_2) = E$. Each constant of motion thus reduces the dimension of the energy surface by one, limiting the available motion to that of a surface of dimension $2n - k$, where k is the number of constants of motion.

To obtain the equations of motion in a trivial way, one is then led to either finding constants of motion from the current Hamiltonian or transforming the coordinates in such a way as to make their discovery trivial. We go down the latter route and are thus led to the theory of canonical transformations.

Canonical transformations are transformations that preserve the Hamiltonian nature of the vector field, or put another way, preserve the form of Hamilton's equations with the new Hamiltonian expressed as a function of the new variables.

A transformation from $(\boldsymbol{q}, \boldsymbol{p})$ to $(\boldsymbol{Q}, \boldsymbol{P})$ is canonical if

$$\dot{Q}^i = \frac{\partial K(\boldsymbol{Q}, \boldsymbol{P})}{\partial P_i}, \tag{2.1.19}$$

$$\dot{P}_i = -\frac{\partial K(\boldsymbol{Q}, \boldsymbol{P})}{\partial Q^i}, \tag{2.1.20}$$

where $K(\boldsymbol{Q}, \boldsymbol{P}) = H(\boldsymbol{q}(\boldsymbol{Q}, \boldsymbol{P}), \boldsymbol{p}(\boldsymbol{Q}, \boldsymbol{P}))$. The canonical transformation is characterised by a generating function G, which allows one to obtain a relationship between the old and new coordinates. A generating function G, is said to be of type 2 if it is a function of the old position and the new momentum coordinates. The relationship between the variables has the following form

$$p_i = \frac{\partial G(\boldsymbol{q}, \boldsymbol{P})}{\partial q^i}, \tag{2.1.21a}$$

$$Q^i = \frac{\partial G(\boldsymbol{q}, \boldsymbol{P})}{\partial P_i}, \tag{2.1.21b}$$

$$K = H + \frac{\partial G}{\partial t}. \tag{2.1.21c}$$

Using (2.1.21a) as the momentum variable in our Hamiltonian we find

$$H\left(\boldsymbol{q}, \frac{\partial G(\boldsymbol{q}, \boldsymbol{P})}{\partial \boldsymbol{q}}\right) = E, \tag{2.1.22}$$

where $\partial G(\boldsymbol{q}, \boldsymbol{P})/\partial \boldsymbol{q} = (\partial G(\boldsymbol{q}, \boldsymbol{P})/\partial q^1, \partial G(\boldsymbol{q}, \boldsymbol{P})/\partial q^2, \cdots, \partial G(\boldsymbol{q}, \boldsymbol{P})/\partial q^n)$. Letting

$$W(\boldsymbol{q}, \boldsymbol{P}, t) := G(\boldsymbol{q}, \boldsymbol{P}) - Et \tag{2.1.23}$$

yields

$$H\left(\boldsymbol{q}, \frac{\partial W}{\partial \boldsymbol{q}}\right) + \frac{\partial W}{\partial t} = 0, \tag{2.1.24}$$

the HJE. Solving the HJE thus allows one to obtain the exact form of the generating function W, which provides a method to change between the old and the new coordinates. We seek a complete solution to the HJE, which is a n-parameter family of surfaces or a solution which contains as many arbitrary constants as independent variables. Since in the HJE equation we have $n + 1$ independent variables we have $n + 1$ arbitrary constants. W enters into the HJE via its derivatives, therefore, one of these arbitrary constants is additive which can be set to zero. Thus, we seek a solution of the form

$$W = S(q^1, q^2, \cdots, q^n, \alpha_1, \alpha_2, \cdots, \alpha_n, t). \tag{2.1.25}$$

We take the new momentum to be the n arbitrary constants $\boldsymbol{\alpha}$. From (2.1.21b) and (2.1.21a) the new coordinates and old momentum are

$$\beta^i = \frac{\partial S}{\partial \alpha_i}, \tag{2.1.26}$$

and

$$p_i = \frac{\partial S}{\partial q^i}, \tag{2.1.27}$$

respectively. Using (2.1.25) in (2.1.24) we obtain the HJE for S

$$H\left(\boldsymbol{q}, \frac{\partial S}{\partial \boldsymbol{q}}\right) + \frac{\partial S}{\partial t} = 0. \tag{2.1.28}$$

Using S as the canonical transformation with (2.1.21c) yields

$$K = H + \frac{\partial S}{\partial t} = 0, \tag{2.1.29}$$

therefore,

$$\dot{Q}^i = 0, \tag{2.1.30}$$

$$\dot{P}_i = 0. \tag{2.1.31}$$

The new coordinates under a canonical transformation, S, are now constants of motion. Using (2.1.26) we can obtain an expression for the n-coordinates $\boldsymbol{q}$ given in terms of the $2n$ arbitrary constants $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ if

$$\det \left. \frac{\partial^2 S(\boldsymbol{q}, \boldsymbol{\alpha})}{\partial \boldsymbol{q} \partial \boldsymbol{\alpha}} \right|_{(\boldsymbol{q}_0, \boldsymbol{\alpha}_0)} \neq 0, \tag{2.1.32}$$

for some $(\boldsymbol{q}_0, \boldsymbol{\alpha}_0) \in \mathbb{R}^n_{\boldsymbol{q}} \times \mathbb{R}^n_{\boldsymbol{\alpha}}$ (a direct product of two n-dimensional coordinate spaces whose points are denoted by $\boldsymbol{q}$ and $\boldsymbol{\alpha}$).

As stated before, since the Hamiltonian, (2.1.17), is independent of time, the energy is conserved and is a constant, say α_1. We also note that since q^1, also does not appear in the Hamiltonian, p_1 is a constant, say α_2. Using (2.1.27), we see that the generating function, S has a linear dependence on q^1. Since p_1 is a constant of motion the generating function can be written as a sum of two functions and an energy-time term

$$S(\boldsymbol{q}, \boldsymbol{\alpha}) = S_2(q^2, \boldsymbol{\alpha}) + \alpha_2 q^1 - \alpha_1 t. \tag{2.1.33}$$

The existence of cyclic coordinates results in a HJE which can be separated into functions of the n-independent variables q^i,

$$S = \sum_{i=1}^{n} S_i(q^i, \boldsymbol{\alpha}). \tag{2.1.34}$$

When (2.1.34) holds the HJE is said to be completely separable. One route to finding constants of motion is to solve the HJE for $\boldsymbol{\alpha}$, and deduce the interpretation from the system's symmetry. Separability is also an indication of the existence of constants of motion which can be shown by considering the following

$$\Phi\left(q^2, q^1, \frac{\partial S}{\partial q^2}, \frac{\partial S}{\partial q^1}\right) = 0. \tag{2.1.35}$$

If this can be separated it can be written in the form

$$\Phi\left(q^2, \frac{\partial S_2}{\partial q^2}, \phi\left(q^1, \frac{\partial S_1}{\partial q^1}\right)\right) = 0 \tag{2.1.36}$$

such that $S = S_1 + S_2$. ϕ can thus be set to a constant

$$\Phi\left(q^2, \frac{\partial S_2}{\partial q^2}, c_1\right) = 0 \tag{2.1.37}$$

$$\phi\left(q^1, \frac{\partial S_1}{\partial q^1}\right) = c_1. \tag{2.1.38}$$

For example, from (2.1.17) a Hamiltonian might be of the form

$$H(q^2, p_1, p_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(q^2), \tag{2.1.39}$$

where V is some continuous function. Thus

$$\phi = \frac{1}{2m}\left(\frac{\partial S_1}{\partial q^1}\right)^2 = c_1, \tag{2.1.40}$$

and

$$\Phi\left(q^2, \frac{\partial S_2}{\partial q^2}, c_1\right) = \frac{c_1^2}{2m} + \frac{p_2^2}{2m} + V(q^2) - E = 0. \tag{2.1.41}$$

Thus, due to the Hamiltonian being separable we are able to obtain a constant of motion

$$p_1 = \frac{\partial S_1}{\partial q^1} = \sqrt{2mc_1} = \alpha_1 \text{ for some } c_1 \in \mathbb{R}. \tag{2.1.42}$$

If the HJE equation is completely separable, (2.1.34), we can obtain n constants of motion. Due to separability, the functions $S_i(q^i, \boldsymbol{\alpha})$ only depend on one of the q^i, but may depend on all the constants $\boldsymbol{\alpha}$. From (2.1.27) and (2.1.34) the momentum p_i, are functions only of the position they are conjugate to, i.e., q^i. Therefore, in the total phase space, the graph $p_i = p_i(q^i)$ can be drawn in the i-th sub-manifold defined by the $2n-2$ equations $q^j =$ const. and $p_j =$ const. with $i \neq j$. The motion is then restricted to each of the i-th sub-manifolds, however, does not represent the full phase space trajectory. The full phase space trajectory involves changing all the position and momentum coordinates simultaneously and thus are not localised to one sub-manifold. We therefore see that with the introduction of n constants of motion, which are identified via separability, the system is completely separable and time evolution can be studied on each sub-manifold. Systems with as many conserved quantities as degrees of freedom are a special class of system and are called integrable. In this thesis, the constants of motion are obtained via cyclic coordinates and thus separability is a direct consequence.

Mathematically the statement of an integrable system is:

Definition 1. *Let N be the $2n$-dimensional phase space of a Hamiltonian system with Hamiltonian function $H(\boldsymbol{q}, \boldsymbol{p})$. Assume that there are n constants of motion, $f_1 = H$, f_2, ..., f_n that are in involution, $\{f_i, f_j\} = 0$, and that are independent, i.e., the n Hamiltonian vector fields generated by f_i are linearly independent and commute with each other, then the Hamiltonian system is said to be integrable.*

Restricting the level surfaces to be compact manifolds again constrains the motion. Systems with this additional constraint are called Liouville integrable.

Theorem 2.1.1. *Let C be the $2n$-dimensional phase space of an integrable Hamiltonian system with Hamiltonian function $H(\boldsymbol{q}, \boldsymbol{p})$. Consider a level* *set of the functions f_i:*

$$\Sigma := \{(\boldsymbol{q}, \boldsymbol{p}) \in C; f_1 = c_1 = const., f_2 = c_1 = const., \cdots, f_n = c_n = const.\} \,. \tag{2.1.43}$$

If Σ is compact then it is homotopic to a torus and the phase flow with Hamiltonian function H determines conditionally periodic motion. The resulting system is coined a Liouville integrable system.

Proof. See [KVA97]. $\square$

As mentioned before, in a completely separable system (2.1.34), each p_i can be written in terms of q^i. If the motion is then confined to a compact level surface (theorem 2.1.1), which is restricted to sub-manifolds of the phase space, the trajectories trace out closed curves labelled as $\mathcal{C}_i$. Since the curves are closed, they are homotopic to circles, $\mathbb{S}^1$. In n-dimensions, we have n closed curves $\mathcal{C}_i$. Thus, a trajectory in phase space is specified by the evolution of the point $(\boldsymbol{q}, \boldsymbol{p}) \in (\mathbb{S}^1)^n = \mathbb{S}^1 \times ... \times \mathbb{S}^1$. The system can thus be described as being confined to the surface of a n-dimensional invariant torus, $\mathbb{T}^n \subset T^*\mathbb{R}^n$. This is called the Liouville-Arnold torus (LAT) and the basis curves of the torus are $\mathcal{C}_i$ with the trajectory in phase space being a combination of the motion on each of the i sub-manifolds which winds around the torus $\mathbb{T}^n$.

2.1.1 Action angle variables

We now seek to define coordinates on the LAT in which any Liouville-integrable system can be expressed. In this sub-section we will not use the Einstein summation.

We know that since the HJE is separable the motion is separated into closed curves $\mathcal{C}_i$, on the n sub-manifolds (q^i, p_i); the level surface is then specified by the n constants of motion $\boldsymbol{\alpha}$ and the angular coordinates are labelled by θ^i.

In general, the $\boldsymbol{\alpha}$ variables are not canonical. Canonical variables can be obtained by considering functions of $\boldsymbol{\alpha}$, $\boldsymbol{J}(\boldsymbol{\alpha}) = (J_1, \cdots, J_n)$, such that the symplectic form becomes

$$\omega^2 = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q^i = \sum_{i=1}^{n} \mathrm{d}J_i \wedge \mathrm{d}\theta^i. \tag{2.1.44}$$

Since the variables $\boldsymbol{J}$ depend on the n-constants $\boldsymbol{\alpha}$, they are constants of motion themselves,

$$\dot{J}_i = -\frac{\partial K}{\partial \theta^i} = 0 \text{ for all } i = 1, \cdots n, \tag{2.1.45}$$

which implies that the new Hamiltonian does not depend on the angle variables i.e.,

$$K(\boldsymbol{J}) = E. \tag{2.1.46}$$

It is convention to normalise the period of the angle to 2π around a closed curve on the torus:

$$\Delta\theta^j = \oint_{\mathcal{C}_j} \mathrm{d}\theta^j = 2\pi. \tag{2.1.47}$$

To construct the canonical transformation from the old variables to the new variables $(\boldsymbol{q}, \boldsymbol{p}) \mapsto (\boldsymbol{J}, \boldsymbol{\theta})$, we look at its type 2 generating function $S(\boldsymbol{q}, \boldsymbol{J}) = \sum_{i=1}^{n} S_i(q^i, \boldsymbol{J})$, along with the relations (2.1.21a), (2.1.21b) and the new Hamiltonian

$$p_i = \frac{\partial S(\boldsymbol{q}, \boldsymbol{J})}{\partial q^i}, \; \theta^i = \frac{\partial S(\boldsymbol{q}, \boldsymbol{J})}{\partial J_i}, \text{ and } K(\boldsymbol{J}) = E. \tag{2.1.48}$$

In the i-th sub-manifold the total derivative (for constant $\boldsymbol{J}$) of S is found to be

$$\mathrm{d}S_i = p_i \mathrm{d}q^i, \tag{2.1.49}$$

where (2.1.48) was used. The generating function then has the form

$$S_i = \int_{q_0^i}^{q^i} p_i \mathrm{d}q^i. \tag{2.1.50}$$

Consider now the change in S_i for a full cycle around the i-th loop, ΔS_i, which encloses area Π_i

$$\Delta S_i = \oint_{\mathcal{C}_i} p_i \mathrm{d}q^i = \Pi_i. \tag{2.1.51}$$

Since $\theta^i = \partial S/\partial J_i = \partial S_i/\partial J_i$ and from (2.1.48), we find

$$2\pi = \oint_{\mathcal{C}_i} \mathrm{d}\theta^i = \oint_{\mathcal{C}_i} \frac{\partial \theta^i}{\partial q^i}\mathrm{d}q^i = \oint_{\mathcal{C}_i} \frac{\partial^2 S_i}{\partial q^i \partial J_j}\mathrm{d}q^i = \frac{\partial}{\partial J_i}\oint_{\mathcal{C}_i} p_i \mathrm{d}q^i = \frac{\partial \Pi_i}{\partial J_i}. \tag{2.1.52}$$

We therefore find $\Pi_i = 2\pi J_i$ and obtain the relation

$$J_i = \frac{1}{2\pi}\oint_{\mathcal{C}_i} p_i \mathrm{d}q^i. \tag{2.1.53}$$

The full generating function, S, is found by first separating the HJE (2.1.28) and obtaining an expression for S in terms of $\boldsymbol{q}$ and $\boldsymbol{\alpha}$. Then using (2.1.48), to find $\boldsymbol{p}$, (2.1.53) can then used to obtain $\boldsymbol{J}$ as functions of $\boldsymbol{\alpha}$. Substituting the expressions for $\boldsymbol{\alpha}$ in terms of $\boldsymbol{J}$ into the generating function, S can then be written in terms of $\boldsymbol{q}$ and $\boldsymbol{J}$. Using (2.1.48) expressions for the old and new variables can be obtained and the full transformation specified.

The equations of motion in the new variables are thus seen to be the solutions of the following

$$\dot{J}_i = 0,\ \dot{\theta}^i = \frac{\partial K(\boldsymbol{J})}{\partial J_i} = \nu^i(\boldsymbol{J}), \tag{2.1.54}$$

for $i = 1, \cdots, n$. The solutions of (2.1.54) are thus found to be

$$J_i(t) = a_i,\ \theta^i(t) = \nu^i(\boldsymbol{a})t + b^i. \tag{2.1.55}$$

Action-angle variables provides a simple way of obtaining the periodic orbits of the system. In the $\boldsymbol{\theta}$-plane, the trajectories are made up of straight lines as can be seen in this simple algebraic manipulation for $n = 2$. From (2.1.55) we have

$$\theta^1(t) = \nu^1(\boldsymbol{a})t + b^1,\ \theta^2(t) = \nu^2(\boldsymbol{a})t + b^2. \tag{2.1.56}$$

Eliminating t yields an equation in terms of $\boldsymbol{\theta}$,

$$\theta^2 = \frac{\nu^2(\boldsymbol{a})}{\nu^1(\boldsymbol{a})}\left(\theta^1 - b^1\right) + b^2. \tag{2.1.57}$$

If $\nu^2(\boldsymbol{a})/\nu^1(\boldsymbol{a}) = m/s \in \mathbb{Q}$, the orbits are closed; put another way, the trajectory winds m times around one axis and s times around another axis, if these are integers, both trajectories will return to the initial point and therefore characterise a periodic orbit. Frequencies which obey this rule are said to be commensurate. For n degrees-of-freedom this condition is stated as

$$\sum_{i=1}^{n} \nu^i k^i = 0 \tag{2.1.58}$$

for $k^i \in \mathbb{Z}$.

As a simple demonstration of HJE we solve a simple case of a particle in the presence of a magnetic field generated by the Landau vector potential which is an integrable system with phase space $N = \mathbb{R}^4$.

To find the Hamiltonian we first obtain the Lagrangian L, of the system for a particle of mass m and charge e in a magnetic field with vector potential $\boldsymbol{A}(\boldsymbol{q})$. This can be found by reducing the Lorentz force law ($c = 1$)

$$m\ddot{\boldsymbol{q}} = e\left(\dot{\boldsymbol{q}} \times \boldsymbol{b} + \boldsymbol{E}\right), \tag{2.1.59}$$

where $\boldsymbol{b} = \nabla \times \boldsymbol{A}$ and $\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t$ (where we have set $\phi = 0$), to have an equivalent form of Lagrange's equations (2.1.1). This can be seen by using

$$\boldsymbol{a} \times (\nabla \times \boldsymbol{z}) = \nabla\left(\boldsymbol{a} \cdot \boldsymbol{z}\right) - \left(\boldsymbol{a} \cdot \nabla\right) \boldsymbol{z} \tag{2.1.60}$$

in (2.1.59) and noting that

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \dot{\boldsymbol{q}} \cdot \nabla \boldsymbol{A} + \frac{\partial \boldsymbol{A}}{\partial t}. \tag{2.1.61}$$

The Lagrangian can then be found by comparing with (2.1.1) which yields

$$L = \frac{1}{2} m \dot{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}} + e \dot{\boldsymbol{q}} \cdot \boldsymbol{A}. \tag{2.1.62}$$

Using

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \tag{2.1.63}$$

we can write $\dot{q}^i$ as a function of $\boldsymbol{p}$ and $\boldsymbol{q}$ and insert this into the expression for L, then using

$$H = \sum_{i=1} \dot{q}^i p_i - L, \tag{2.1.64}$$

the Legendre transform of L, we obtain $H(\boldsymbol{q}, \boldsymbol{p})$. Using the above procedure we obtain for the case of no electric field the following Hamiltonian

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2m} \langle \boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{q}), \boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{q}) \rangle_{\mathbb{R}^{2n}} \tag{2.1.65}$$

where the magnetic field strength is

$$\boldsymbol{b} = \nabla \times \boldsymbol{A}. \tag{2.1.66}$$

The vector potential for the Landau gauge has the following form, $\boldsymbol{A}(q^1, q^2) = (-bq^2, 0)$ and the corresponding Hamiltonian

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}(p_1 + bq^2)^2 + \frac{p_2^2}{2}, \tag{2.1.67}$$

where $m = c = e = 1$. The time t, and the first position coordinate q^1, do not appear in the above Hamiltonian therefore the energy E and the the first momentum, p_1, are constants of motion. Since $\{H, p_1\} = 0$ we have two constants of motion in involution. This system has 2 degrees-of-freedom and by definition 1, the Hamiltonian is integrable. The system is not however Liouville-integrable since the level set defined by $p_1 = c_2$ is not compact.

Using (2.1.28) we obtain the HJE

$$\frac{1}{2}\left(\frac{\partial S}{\partial q^1}+bq^2\right)^2+\frac{1}{2}\left(\frac{\partial S}{\partial q^2}\right)^2=\alpha_1. \tag{2.1.68}$$

Separating yields

$$\frac{\partial S}{\partial q^1}=-bq^2\pm\sqrt{2\alpha_1-\left(\frac{\partial S}{\partial q^2}\right)^2}. \tag{2.1.69}$$

Using the separation of variables for partial differential equations and the function $S=X_1(q^1,\boldsymbol{\alpha})+X_2(q^2,\boldsymbol{\alpha})$ we find

$$\begin{aligned}\frac{\mathrm{d}X_1}{\mathrm{d}q^1}&=\alpha_2,\ \text{and}\\ \alpha_2&=-bq^2\pm\sqrt{2\alpha_1-\left(\frac{\mathrm{d}X_2}{\mathrm{d}q^2}\right)^2}.\end{aligned} \tag{2.1.70}$$

Solving these for the X_i we thus obtain

$$\begin{aligned}X_1&=\alpha_2q^1,\\ X_2&=\pm\int\sqrt{2\alpha_1-(bq^2+\alpha_2)^2}\mathrm{d}q^2,\end{aligned} \tag{2.1.71}$$

where

$$\begin{aligned}p_1&=\alpha_2,\\ p_2&=\pm\sqrt{2\alpha_1-(bq^2+\alpha_2)^2},\end{aligned} \tag{2.1.72}$$

with

$$S=\alpha_2q^1\pm\int\sqrt{2\alpha_1-(bq^2+\alpha_2)^2}\mathrm{d}q^2-\alpha_1t. \tag{2.1.73}$$

Using (2.1.21b) we find the expressions for $\boldsymbol{Q}=\boldsymbol{\beta}$ to be of the form

$$\begin{aligned}\beta^1&=\pm\int\frac{1}{\sqrt{2\alpha_1-(bq^2+\alpha_2)^2}}\mathrm{d}q^2-t\\ &=\pm\frac{1}{b}\arcsin\left(\frac{\alpha_2+bq^2}{\sqrt{2\alpha_1}}\right)-t,\\ \beta^2&=\mp\int\frac{bq^2+\alpha_2}{\sqrt{2\alpha_1-(bq^2+\alpha_2)^2}}\mathrm{d}q^2+q^1\\ &=\pm\frac{1}{b}\sqrt{2\alpha_1-(bq^2+\alpha_2)^2}+q^1.\end{aligned} \tag{2.1.74}$$

We thus find that the equations of motion in the configuration space are

$$\begin{aligned} \alpha_2 + bq^2 &= \mp\sqrt{2\alpha_1}\sin\bigl(b(\beta^1 + t)\bigr), \\ \beta^2 - q^1 &= \pm\frac{1}{b}\sqrt{2\alpha_1}\cos\bigl(b(\beta^1 + t)\bigr), \end{aligned} \tag{2.1.75}$$

therefore

$$\left(\frac{\alpha_2}{b} + q^2\right)^2 + (\beta^2 - q^1)^2 = \frac{2\alpha_1}{b}. \tag{2.1.76}$$

The trajectories on the configuration manifold are thus ellipses centred at $(\beta^2, -\frac{\alpha_2}{b})$. We will treat the Liouville-integrable case when we restrict the phase space to $\mathbb{T}^{2n}$.

Chapter 3

Weyl quantisation

In this chapter we review the theory of Weyl quantisation on $\mathbb{R}^n$ by developing a unitary representation of the Heisenberg group. Then, discussing the various conclusions of restricting the position and momentum space to the torus we obtain a phase space quantisation condition. Using this, we extend Weyl quantisation to the torus. We then study the magnetic Weyl calculus by generalising the unitary representation of the Heisenberg group to include a vector potential. Discretising the so called magnetic translation operators we arrive at the magnetic Weyl calculus on the torus.

3.1 Quantum mechanics

Before discussing Weyl quantisation, it is helpful to introduce some basic definitions and theorems about quantum mechanics which will be useful later on in the thesis. The main body of the material was taken from [ME14, Mar14, Gie00, Zwo12] and, [MM04].

We will denote by $\mathcal{H}$, a Hilbert space with an inner product $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$,

$$(\psi, \phi) \mapsto \langle \psi, \phi \rangle_{\mathcal{H}},$$

which is conjugate linear in its first argument. The Hilbert space for a quantum system with configuration space $\mathbb{R}^n$ is defined as the following.

Definition 2. *The Hilbert space* $\mathcal{H} = L^2(\mathbb{R}^n)$ *is the set of complex-valued measurable functions on* $\mathbb{R}^n$ *such that they are square-integrable with respect to the Lebesgue measure* $\mathrm{d}\boldsymbol{x} := \mathrm{d}x^1\mathrm{d}x^2 \cdots \mathrm{d}x^n$ *i.e.,*

$$\int_M |\psi(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} < \infty,$$

with the inner product between two function $\psi, \phi \in L^2(\mathbb{R}^n)$ *to be defined as* $\langle \cdot, \cdot \rangle_{L^2(M)} : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to \mathbb{C}$,

$$\langle \psi, \phi \rangle_{L^2(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \psi^*(\boldsymbol{x})\phi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x},$$

where $*$ *denotes the complex conjugate.*

An operator $\mathrm{Op}[A]$ acting on a vector in $\psi(\boldsymbol{q}) \in \mathcal{H} = L^2(\mathbb{R}^n)$, can be written in the form of a kernel

$$\left(\mathrm{Op}[A]\psi\right)(\boldsymbol{q}) = \int_{\mathbb{R}^n} K_A\left(\boldsymbol{q}, \boldsymbol{z}\right) \psi(\boldsymbol{z}) \, \mathrm{d}\boldsymbol{z}. \tag{3.1.1}$$

Observables which are studied in quantum mechanics are characterised by self-adjoint operators. In this thesis the operators of interest are bounded operators, with this in mind we present some definitions and theorems for bounded operators $\mathrm{Op}[A] : \mathcal{H} \to \mathcal{H}$ (see [Ree80] for a complete exposition).

Definition 3. *A bounded operator* $\mathrm{Op}[A]$ *on* $\mathcal{H}$ *is Hermitian (symmetric) if*

$$\langle \psi, \mathrm{Op}[A]\phi \rangle_{\mathcal{H}} = \langle \mathrm{Op}[A]\psi, \phi \rangle_{\mathcal{H}}$$

for all $\psi, \phi \in \mathcal{H}$

Definition 4. *For a bounded linear operator* $\mathrm{Op}[A] : \mathcal{H} \to \mathcal{H}$, *we define the adjoint* $\mathrm{Op}[A]^\dagger : \mathcal{H} \to \mathcal{H}$ *by the formula*

$$\langle \phi, \mathrm{Op}[A]\psi \rangle_{\mathcal{H}} = \langle \mathrm{Op}[A]^\dagger\phi, \psi \rangle_{\mathcal{H}}.$$

A bounded operator Op$[A]$, *is self-adjoint if it is Hermitian.*

Quantisation of a classical observable is written in terms of an integral kernel, as such, we provide an important proposition on the self-adjointness for operators of this form.

Proposition 3.1.1. *An operator* $\mathrm{Op}[A] : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *can be written in the form of an integral kernel,*

$$(\mathrm{Op}[A]\psi)(\boldsymbol{q}) := \int_{\mathbb{R}^n} K_A(\boldsymbol{q}, \boldsymbol{z})\,\psi(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}. \tag{3.1.2}$$

The adjoint, $\mathrm{Op}[A]^\dagger$, *has kernel* $K_A^*(\boldsymbol{z}, \boldsymbol{q})$. *If* $\mathrm{Op}[A]$ *is bounded then it is self-adjoint if*

$$K_A(\boldsymbol{q}, \boldsymbol{z}) = K_A^*(\boldsymbol{z}, \boldsymbol{q}). \tag{3.1.3}$$

Proof. From definition 2 we see that by Fubini's theorem

$$\begin{aligned}
\langle \phi, \mathrm{Op}[A]\psi \rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi^*(\boldsymbol{q}) K_A(\boldsymbol{z}, \boldsymbol{q})\,\psi(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{q} \\
&= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} K_A(\boldsymbol{q}, \boldsymbol{z})\,\phi^*(\boldsymbol{q})\,\mathrm{d}\boldsymbol{q}\right)\psi(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} \\
&= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} K_A^*(\boldsymbol{q}, \boldsymbol{z})\,\phi(\boldsymbol{q})\,\mathrm{d}\boldsymbol{q}\right)^*\psi(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} \\
&= \langle \mathrm{Op}[A]^\dagger\phi, \psi \rangle_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{3.1.4}$$

If $\mathrm{Op}[A]$ is bounded then the adjoint is defined via definition 3 and therefore is self adjoint if $K_A(\boldsymbol{z}, \boldsymbol{q}) = K_A^*(\boldsymbol{q}, \boldsymbol{z})$. □

Since our phase space is compact our Hilbert space is finite dimensional, we will thus take as our Hilbert space $\mathcal{H}_{N^n} = \left(\mathbb{C}^{N^n}, \langle\cdot,\cdot\rangle_{\mathbb{C}^{N^n}}\right)$ where $\mathbb{C}^{N^n}$ is the complex vector space of dimension N^n and

$$\langle \phi, \psi \rangle_{\mathbb{C}^{N^n}} := \sum_{i=0}^{N^n-1} \phi_i^* \psi_i. \tag{3.1.5}$$

Observables, $\mathrm{Op}[A]$ on a finite dimensional Hilbert space are represented as elements of the space of $N^n \times N^n$ complex valued matrices, $\mathcal{M}_{N^n}(\mathbb{C})$. In this case they are bounded and thus the labels self-adjoint and Hermitian are synonymous.

Definition 5. *The matrix elements of an operator,* $\mathrm{Op}[A] \in \mathcal{M}_{N^n}(\mathbb{C})$ *is defined as*

$$(\mathrm{Op}[A]\psi)_i = \sum_{j=0}^{N^n-1} (\mathrm{Op}[A])_{ij}\, \psi_j, \tag{3.1.6}$$

where $(\mathrm{Op}[A])_{ij}$ *denotes the* ij*-th element of the matrix* $\mathrm{Op}[A]$.

Definition 6. *The adjoint of a finite dimensional operator* $\mathrm{Op}[A]$ *with matrix elements* $(\mathrm{Op}[A])_{ij}$ *is defined as*

$$\left(\mathrm{Op}[A]^\dagger\right)_{ij} = \mathrm{Op}[A]^*_{ji}. \tag{3.1.7}$$

The observables for a quantum system are represented by Hermitian operators, $\mathrm{Op}[A]$, on the Hilbert space $\mathcal{H}$, where $\mathrm{Op}[A]$ will be the quantisation of some function A on N (for the class of functions which we will be studying in this thesis, the quantisation yields bounded operators on $L^2(\mathbb{R}^n)$). The spectrum, $\mathrm{Spec}(\mathrm{Op}[A])$, of $\mathrm{Op}[A]$ are the definite values which the states in $\mathcal{H}$ obtain after the measurement of $\mathrm{Op}[A]$ and are the eigenvalues of $\mathrm{Op}[A]$.

Definition 7. *Let* $\mathrm{Op}[A]$ *be a linear operator acting on* $\psi \in \mathcal{H}$*. When* $\dim\mathcal{H} < \infty$ *then an eigenvalue is defined as a solution of*

$$\mathrm{Op}[A]\psi = \lambda\psi,$$

such that $\psi \neq 0$*. An eigenvalue* λ *is said to be an element of the spectrum* *Spec* $(\mathrm{Op}[A])$.

We now prove the elementary theorem that the eigenvalues of finite dimensional Hermitian operators are real.

Theorem 3.1.1. *The eigenvalues of a finite dimensional Hermitian operator are real.*

Proof. Using definition 3 and 7, we find

$$\begin{aligned}\langle\psi, \mathrm{Op}[A]\psi\rangle_{\mathcal{H}} &= \langle\psi, \lambda\psi\rangle_{\mathcal{H}} = \lambda\langle\psi,\psi\rangle_{\mathcal{H}} \\ &= \langle \mathrm{Op}[A]\psi, \psi\rangle_{\mathcal{H}} = \langle\lambda\psi,\psi\rangle_{\mathcal{H}} = \lambda^*\langle\psi,\psi\rangle_{\mathcal{H}}. \end{aligned} \tag{3.1.8}$$

For $\psi \neq 0$ we find that $\lambda = \lambda^*$ and therefore, $\lambda \in \mathbb{R}$. □

Finding the spectrum of quantum observables in a finite dimensional Hilbert space can now be stated as a problem in linear algebra. To find the eigenvalues of a finite dimensional operator, $\mathrm{Op}[A]$, one solves the equation $\det(\mathrm{Op}[A] - \lambda I) = 0$, where I is the $n \times n$ identity operator. In the case of large matrices, a suitable diagonalisation algorithm can be implemented numerically to obtain the eigenvalues. The cardinality of $\mathrm{Spec}(\mathrm{Op}[A])$ is then defined to be the dimension of the operator.

3.2 Non-magnetic Weyl quantisation

Having discussed the basics of operators we are now in a position to develop a suitable quantisation scheme from which classical observables on phase space are represented as self-adjoint operators in some Hilbert space.

3.2.1 Weyl quantisation on $\mathbb{R}^{2n}$

Weyl quantisation associates to a continuous function

$$f \in \mathscr{S}(\mathbb{R}^{2n}) \tag{3.2.1}$$

$$:= \left\{ f \in C^\infty(\mathbb{R}^{2n});\ \sup_{\boldsymbol{x}\in\mathbb{R}^{2n}} |\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{\boldsymbol{x}}^{\boldsymbol{\beta}} f| < \infty,\ \text{for all multiindices } \boldsymbol{\alpha}, \boldsymbol{\beta} \right\}, \tag{3.2.2}$$

(where

$$\boldsymbol{x}^{\boldsymbol{\alpha}} := (x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n} \text{ for } \boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) \tag{3.2.3}$$

and

$$\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}} := \partial_{x^1}^{\beta_1} \cdots \partial_{x^n}^{\beta_n} \text{ for } \boldsymbol{\beta} = (\beta_1, \cdots, \beta_n), \tag{3.2.4}$$

with $\partial_j = \partial/\partial x^j$), an operator which is expressed as an expansion in terms of translation operators on $L^2(\mathbb{R}^n)$ that belong to the unitary irreducible representations of the Heisenberg group.

In the previous chapter we noted the existence of the symplectic form which gives rise to a volume element in phase space and is used in the definition of the Heisenberg group. The derivation of the following shadows that of [Fol89].

One first defines the Heisenberg Lie algebra $\mathfrak{h}$, by considering the vector space $\mathbb{R}^{2n+1}$ with coordinates $(p_1, ..., p_n, q^1, ..., q^n, t) = (\boldsymbol{p}, \boldsymbol{q}, t)$ and a Lie bracket defined as

$$[(\boldsymbol{p}, \boldsymbol{q}, t), (\boldsymbol{l}, \boldsymbol{y}, t')] = (0, 0, -\sigma((\boldsymbol{q}, \boldsymbol{p}), (\boldsymbol{y}, \boldsymbol{l}))), \tag{3.2.5}$$

where $\sigma : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ is the symplectic form and $(l_1, ..., l_n, y^1, ..., y^n, t') = (\boldsymbol{l}, \boldsymbol{y}, t')$. One sees by multiple applications of the Lie bracket (3.2.5), the Jacobi identity is satisfied and that indeed, the Lie bracket does make $\mathbb{R}^{2n+1}$ a Lie algebra. If one picks the standard basis for $\mathbb{R}^{2n+1}$ then the Lie algebra structure is given by

$$[p_j, p_k] = [q^j, q^k] = [p_j, t] = [q^j, t] = 0, \text{ and } [p_j, q^k] = \delta_j^k t. \tag{3.2.6}$$

Therefore we see that the Poisson brackets with respect to $\boldsymbol{x}$ and $\boldsymbol{\xi}$,

$$\{\xi_j, \xi_k\}_{x,\xi} = \{x^j, x^k\}_{x,\xi} = 0, \ \{\xi_j, x^k\}_{x,\xi} = \delta_j^k, \tag{3.2.7}$$

and the quantum commutation relations

$$\left[\hat{P}_j, \hat{P}_k\right] = \left[\hat{Q}^j, \hat{Q}^k\right] = 0, \ \left[\hat{Q}^k, \hat{P}_j\right] = i\hbar\delta_j^k I, \tag{3.2.8}$$

span a Lie algebra which is isomorphic to $\mathfrak{h}$. To identify the Lie group corresponding to $\mathfrak{h}$, a suitable matrix representation is chosen and exponentiated. We define $m(\boldsymbol{p}, \boldsymbol{q}, t) \in M_{2n+1}(\mathbb{R})$ to be

$$m(\boldsymbol{p}, \boldsymbol{q}, t) = \begin{pmatrix} 0 & q^1 & \cdots & q^n & t \\ 0 & 0 & \cdots & 0 & p_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & p_n \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{3.2.9}$$

where $\mathbb{M}_{2n+1}(\mathbb{R})$ is the space of real $2n+1 \times 2n+1$ matrices. We therefore find that

$$[m(\boldsymbol{p}, \boldsymbol{q}, t), m(\boldsymbol{l}, \boldsymbol{y}, t')] = m(\boldsymbol{0}, \boldsymbol{0}, -\sigma((\boldsymbol{q}, \boldsymbol{p}), (\boldsymbol{y}, \boldsymbol{l}))), \tag{3.2.10}$$

under matrix multiplication. We note that

$$\exp m(\boldsymbol{p}, \boldsymbol{q}, t) \exp m(\boldsymbol{l}, \boldsymbol{y}, t) = \exp m \left(\boldsymbol{p} + \boldsymbol{l}, \boldsymbol{q} + \boldsymbol{y}, t + t' - \frac{1}{2}\sigma((\boldsymbol{q}, \boldsymbol{p}), (\boldsymbol{y}, \boldsymbol{l})) \right), \tag{3.2.11}$$

and with the association of an element in $(\boldsymbol{p}, \boldsymbol{q}, t) \mapsto \exp m(\boldsymbol{p}, \boldsymbol{q}, t)$, we therefore obtain the Heisenberg group $\mathbb{H}_{2n+1}(\mathbb{R})$, with group law $(\boldsymbol{p}, \boldsymbol{q}, t) \circ (\boldsymbol{y}, \boldsymbol{l}, t') = (\boldsymbol{q} + \boldsymbol{y}, \boldsymbol{p} + \boldsymbol{l}, t + t' - \frac{1}{2}(\langle \boldsymbol{y}, \boldsymbol{p} \rangle_{\mathbb{R}^2} - \langle \boldsymbol{q}, \boldsymbol{l} \rangle_{\mathbb{R}^2}))$.

We now look for a map from the Heisenberg algebra to the space of skew-Hermitian operators, $\mathcal{S}_H$ such that the map is a Lie algebra homomorphism. We then exponentiate this expression to obtain a unitary representation of the Heisenberg group.

Since $\hat{Q}^k$ and $\hat{P}_j$, obey the commutation relations a map from $\mathfrak{h}$ to $\mathcal{S}_H$ can be found. We define this map to be

$$\begin{aligned} d\rho_\hbar(\boldsymbol{p}, \boldsymbol{q}, t) &= \frac{i}{\hbar} \left(\boldsymbol{p} \cdot \hat{\boldsymbol{Q}} - \boldsymbol{q} \cdot \hat{\boldsymbol{P}} + tI \right) \\ &= i\frac{\boldsymbol{p}}{\hbar} \cdot \hat{\boldsymbol{Q}} - \boldsymbol{p} \cdot \partial_{\boldsymbol{x}} + i\frac{t}{\hbar} I. \end{aligned} \tag{3.2.12}$$

To be a Lie algebra homomorphism $d\rho_\hbar$ is to satisfy

$$d\rho_\hbar \left([(\boldsymbol{p}, \boldsymbol{q}, t), (\boldsymbol{l}, \boldsymbol{y}, t')]_{\mathfrak{h}} \right) = [d\rho_\hbar(\boldsymbol{p}, \boldsymbol{q}, t), d\rho_\hbar(\boldsymbol{l}, \boldsymbol{y}, t)]_{\mathcal{S}_H}. \tag{3.2.13}$$

Using (3.2.5) and (3.2.12) the left hand side is found to be

$$d\rho_\hbar\left([(\boldsymbol{p},\boldsymbol{q},t),(\boldsymbol{l},\boldsymbol{y},t')]_{\mathfrak{h}}\right) = -\frac{i}{\hbar}(\langle \boldsymbol{y},\boldsymbol{p}\rangle_{\mathbb{R}^2} - \langle \boldsymbol{q},\boldsymbol{l}\rangle_{\mathbb{R}^2})I. \tag{3.2.14}$$

The right hand side is found by using (3.2.12) and (3.2.8) we find

$$\begin{aligned}
&[d\rho_\hbar(\boldsymbol{p},\boldsymbol{q},t), d\rho_\hbar(\boldsymbol{l},\boldsymbol{y},t)]_{\mathcal{S}_H} \\
&= \left[\frac{i}{\hbar}\left(\boldsymbol{p}\cdot\hat{\boldsymbol{Q}} - \boldsymbol{q}\cdot\hat{\boldsymbol{P}} + tI\right), \frac{i}{\hbar}\left(\boldsymbol{l}\cdot\hat{\boldsymbol{Q}} - \boldsymbol{y}\cdot\hat{\boldsymbol{P}} + t'I\right)\right] \\
&= -\frac{i}{\hbar}\left(p_1 y^1 + \cdots + p_n y^n\right) I + \frac{i}{\hbar}(q_1 l^1 + \cdots + q_n l^n) I \\
&= -\frac{i}{\hbar}\left(\langle \boldsymbol{p},\boldsymbol{y}\rangle_{\mathbb{R}^2} - \langle \boldsymbol{q},\boldsymbol{l}\rangle_{\mathbb{R}^2}\right) I.
\end{aligned} \tag{3.2.15}$$

Exponentiating $d\rho(\boldsymbol{p},\boldsymbol{q},t)$, leads to a unitary representation of the Heisenberg group, $\mathbb{H}_{2n+1}(\mathbb{R})$. We now wish to find the action of this representation on an element of $L^2(\mathbb{R}^n)$. This can be shown by letting

$$g(\boldsymbol{q},s) := \left(\mathrm{e}^{-s\boldsymbol{q}\cdot\partial_{\boldsymbol{x}} + is\frac{\boldsymbol{p}}{\hbar}\cdot\hat{\boldsymbol{Q}}} f\right)(\boldsymbol{x}), \tag{3.2.16}$$

which is a solution to $\partial g/\partial s = (-\boldsymbol{q}\cdot\partial_{\boldsymbol{x}} + i\frac{\boldsymbol{p}}{\hbar}\cdot\hat{\boldsymbol{Q}})g$, where $g(\boldsymbol{x},0) = f(\boldsymbol{x})$. We therefore have

$$\frac{\partial g}{\partial s} + \langle \boldsymbol{q}, \partial_{\boldsymbol{x}} g\rangle = i\frac{\langle \boldsymbol{x},\boldsymbol{p}\rangle}{\hbar} g. \tag{3.2.17}$$

Setting $G(s) = g(\boldsymbol{x}(s),s) = g(\boldsymbol{x}+\boldsymbol{q}s, s)$, (3.2.17) now reduces to

$$G'(s) = \frac{i}{\hbar}\langle \boldsymbol{x}+\boldsymbol{q}s, \boldsymbol{p}\rangle G(s), \tag{3.2.18}$$

and can be easily solved. Letting $s=1$ and $\boldsymbol{x}\mapsto\boldsymbol{x}-\boldsymbol{q}$ in the solution we find

$$g(\boldsymbol{x}) = \rho(\boldsymbol{p},\boldsymbol{q})f(\boldsymbol{x}) = \left(\mathrm{e}^{\frac{i}{\hbar}\boldsymbol{p}\cdot\hat{\boldsymbol{Q}} - \frac{i}{\hbar}\boldsymbol{q}\cdot\hat{\boldsymbol{P}}} f\right)(\boldsymbol{x}) = \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{p},\boldsymbol{x}\rangle - \frac{i}{2\hbar}\langle \boldsymbol{p},\boldsymbol{q}\rangle} f(\boldsymbol{x}-\boldsymbol{q}). \tag{3.2.19}$$

We therefore write the representation of the Heisenberg group acting on a vector $f(\boldsymbol{x})\in L^2(\mathbb{R}^n)$ as

$$\left(U(\boldsymbol{p},\boldsymbol{q},t)f\right)(\boldsymbol{x}) = \mathrm{e}^{\frac{i}{\hbar}t}\left(\rho_\hbar(\boldsymbol{p},\boldsymbol{q})f\right)(\boldsymbol{x}) = \mathrm{e}^{\frac{i}{\hbar}t}\mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{p},\boldsymbol{x}\rangle - \frac{i}{2\hbar}\langle \boldsymbol{p},\boldsymbol{q}\rangle} f(\boldsymbol{x}-\boldsymbol{q}). \tag{3.2.20}$$

In what follows we set $t = 0$ and define $U(\boldsymbol{p}, \boldsymbol{q}, 0) := U(\boldsymbol{p}, \boldsymbol{q})$. The group law asserts that

$$U(\boldsymbol{p}, \boldsymbol{q})U(\boldsymbol{l}, \boldsymbol{y}) = U\left(\boldsymbol{p} + \boldsymbol{l}, \boldsymbol{q} + \boldsymbol{y}, -\frac{1}{2}\sigma((\boldsymbol{q}, \boldsymbol{p}), (\boldsymbol{y}, \boldsymbol{l}))\right). \tag{3.2.21}$$

The Schrödinger representation, $\rho_\hbar(\boldsymbol{p}, \boldsymbol{q})$, is a unitary representation of $\mathbb{H}_{2n+1}(\mathbb{R})$ that acts on $L^2(\mathbb{R}^n)$. By the Stone-von Neumann theorem, it is the only unitary representation up to unitary equivalences, labeled by $\hbar$, of $\mathbb{H}_{2n+1}(\mathbb{R})$ (see [ME14]).

Weyl quantisation is the association of a function $\exp\left[\frac{i}{\hbar}(\langle \boldsymbol{p}, \boldsymbol{x}\rangle - \langle \boldsymbol{q}, \boldsymbol{\xi}\rangle)\right]$ with the corresponding operator $\exp\left[\frac{i}{\hbar}(\boldsymbol{p} \cdot \hat{\boldsymbol{Q}} - \boldsymbol{q} \cdot \hat{\boldsymbol{P}})\right]$. With this identification one is able to expand a continuous function in terms of exponentials by the symplectic Fourier transform, $\mathcal{F}_\hbar$. The Weyl quantisation of a symbol $f(\boldsymbol{x}, \boldsymbol{\xi})$ is defined as

$$\mathrm{Op}_\hbar^W[f] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{q}\, \mathrm{d}\boldsymbol{p}\, (\mathcal{F}_\hbar f)(\boldsymbol{q}, \boldsymbol{p}) U(\boldsymbol{p}, \boldsymbol{q}), \tag{3.2.22}$$

where

$$(\mathcal{F}_\hbar f)(\boldsymbol{q}, \boldsymbol{p}) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{l}\, \mathrm{e}^{-\frac{i}{\hbar}\sigma((\boldsymbol{q},\boldsymbol{p}),(\boldsymbol{y},\boldsymbol{l}))} f(\boldsymbol{y}, \boldsymbol{l}). \tag{3.2.23}$$

Letting $\int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{\eta} = \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{q} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{p} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{y} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{l}$, the Weyl quantisation of a symbol, $f(\boldsymbol{q}, \boldsymbol{q}) \in C^\infty(\mathbb{R}^n)$ is found to be

$$\begin{aligned}\left(\mathrm{Op}_\hbar^W[f]\psi\right)(\boldsymbol{z}) &= \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{\eta}\, \mathrm{e}^{-\frac{i}{\hbar}\langle \boldsymbol{y},\boldsymbol{p}\rangle + \frac{i}{\hbar}\langle \boldsymbol{l},\boldsymbol{q}\rangle} f(\boldsymbol{y}, \boldsymbol{l}) \mathrm{e}^{\frac{i}{\hbar}(\langle \boldsymbol{p},\boldsymbol{z}\rangle - \frac{1}{2}\langle \boldsymbol{p},\boldsymbol{q}\rangle)}\psi(\boldsymbol{z} - \boldsymbol{q}) \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{\eta}'\delta\left(\boldsymbol{z} - \boldsymbol{y} - \frac{1}{2}\boldsymbol{q}\right)\mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{q},\boldsymbol{l}\rangle} f(\boldsymbol{y}, \boldsymbol{l})\psi(\boldsymbol{z} - \boldsymbol{q}) \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{q} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{l}\, \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{q},\boldsymbol{l}\rangle} f\left(\boldsymbol{z} - \frac{1}{2}\boldsymbol{q}, \boldsymbol{l}\right)\psi(\boldsymbol{z} - \boldsymbol{q}),\end{aligned} \tag{3.2.24}$$

where $\int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{\eta}' = \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{q} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{y} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{l}$. Letting $\boldsymbol{w} = \boldsymbol{z} - \boldsymbol{q}$ yields

$$\left(\mathrm{Op}_\hbar^W[f]\psi\right)(\boldsymbol{z}) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{w} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{l}\, \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{z}-\boldsymbol{w},\boldsymbol{l}\rangle} f\left(\frac{\boldsymbol{z} + \boldsymbol{w}}{2}, \boldsymbol{l}\right)\psi(\boldsymbol{w}). \tag{3.2.25}$$

3.2.2 Weyl quantisation on $\mathbb{T}^{2n}$

In this thesis we restrict the phase space to that of the torus. The classical symbols we therefore wish to quantise are periodic functions, $f(\boldsymbol{q}+\ell_q, \boldsymbol{p}+\ell_p) = f(\boldsymbol{q}, \boldsymbol{p})$, where it is understood that $\boldsymbol{a}+b = (a_1+b, a_2+b, ..., a_n+b)$. A function with this property can be expanded as a Fourier series of the form

$$f(\boldsymbol{q}, \boldsymbol{p}) = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}} f_{\boldsymbol{n},\boldsymbol{m}} \mathrm{e}^{2\pi i\left(\frac{\langle \boldsymbol{n},\boldsymbol{q}\rangle}{\ell_q} - \frac{\langle \boldsymbol{m},\boldsymbol{p}\rangle}{\ell_p}\right)}, \tag{3.2.26}$$

where

$$f_{\boldsymbol{n},\boldsymbol{m}} = \frac{1}{\ell_q^n \ell_p^n} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{q}\, \mathrm{d}\boldsymbol{p}\, \mathrm{e}^{-2\pi i\left(\frac{\langle \boldsymbol{m},\boldsymbol{p}\rangle}{\ell_p} - \frac{\langle \boldsymbol{n},\boldsymbol{q}\rangle}{\ell_q}\right)} f(\boldsymbol{q}, \boldsymbol{p}). \tag{3.2.27}$$

One can speculate that the phase space translation operators obey the periodic nature of the phase space, this means on an operator level, that the phase space translation operators commute

$$[U(\ell_p, 0), U(0, \ell_q)] = 0. \tag{3.2.28}$$

Using (3.2.21) we obtain n conditions of the form

$$U(\ell_p, 0)U(0, \ell_q) = \mathrm{e}^{-\frac{i\ell_q\ell_p}{\hbar}} U(0, \ell_q)U(\ell_p, 0), \tag{3.2.29}$$

if the operators commutes, there exists $N \in \mathbb{N}$ such that

$$\ell_q\ell_p = 2\pi\hbar N. \tag{3.2.30}$$

Restricting the phase space to that of a torus and requiring that the translation operators commute has thus given us a phase space quantisation condition which is a relation between the value of Planck's constant $\hbar$, and the area of $\mathbb{T}^2 = \mathbb{R}^2/(\ell_q\mathbb{Z} \oplus \ell_p\mathbb{Z})$.

The semiclassical limit is defined as taking the limit as $\hbar \to 0$. The phase space quantisation condition gives an equivalent limit so as to avoid letting a fundamental constant tend to zero. Thus, the semiclassical limit is equivalent to $N \to \infty$. We will from now on assume (3.2.30) holds throughout the remainder of the thesis. By restricting the phase space to a torus, a quantisation condition was obtained which can be used to study different equivalent limits, [JBK17]. The torus also restricts the available states in the Hilbert space.

We expect the quantum states to have same periodicity as the classical symbols, i.e.,

$$U(\ell_p, 0)\psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) \tag{3.2.31}$$

$$U(0, \ell_q)\psi(\boldsymbol{x}) = \psi(\boldsymbol{x}). \tag{3.2.32}$$

Looking at (3.2.31) and (3.2.20) we find

$$\left(\mathrm{e}^{\frac{i}{\hbar}\ell_p x} - 1\right)\psi(\boldsymbol{x}) = 0. \tag{3.2.33}$$

Solving this in the distributional sense i.e.,

$$\int_{\mathbb{R}^n} \left(\mathrm{e}^{\frac{i}{\hbar}\ell_p x} - 1\right)\psi(\boldsymbol{x})\eta(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0, \tag{3.2.34}$$

where $\eta(\boldsymbol{x})$ is some suitably chosen smooth, rapid decaying and compactly supported test function. We find as a solution a series of Dirac deltas of the form

$$\psi(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^n} c_{\boldsymbol{n}}\delta\left(\boldsymbol{x} + \frac{2\pi\hbar}{\ell_p}\boldsymbol{n}\right). \tag{3.2.35}$$

Applying (3.2.32) with (3.2.20) again, yields

$$U(0, \ell_q)\psi(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^n} c_{\boldsymbol{n}}\delta\left(\boldsymbol{x} - \ell_q + \frac{2\pi\hbar}{\ell_p}\boldsymbol{n}\right) \tag{3.2.36}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^n} c_{\boldsymbol{n}}\delta\left(\boldsymbol{x} + \frac{2\pi\hbar}{\ell_p}\left(-\frac{\ell_q\ell_p}{2\pi\hbar} + \boldsymbol{n}\right)\right) \tag{3.2.37}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^n} c_{\boldsymbol{n}}\delta\left(\boldsymbol{x} + \frac{2\pi\hbar}{\ell_p}(-N + \boldsymbol{n})\right). \tag{3.2.38}$$

Shifting the index by N in each argument of $c_{\boldsymbol{n}}$, we therefore see that $c_{n_1+N,n_2+N,\dots,n_n+N} = c_{n_1,n_2,\dots,n_n}$. Defining

$$\boldsymbol{e_j}(\boldsymbol{x}) = \sum_{\boldsymbol{n}} \delta\left(\boldsymbol{x} + \boldsymbol{j}\frac{\ell_p}{N} + \boldsymbol{n}\ell_p\right), \tag{3.2.39}$$

which forms a basis of $L^2(\mathbb{T}^n)$, i.e., $\psi(\boldsymbol{x}) = \sum_{\boldsymbol{j}\in\mathbb{Z}_N^n} \psi_{\boldsymbol{j}} e_{\boldsymbol{j}}(\boldsymbol{x})$ where $\mathbb{Z}_N^n := \{0,1,2,\dots,N-1\}^n$, allows one to set up an identification of vectors in $L^2(\mathbb{T}^n)$ with vectors in $\mathbb{C}^{N^n}$ which we label as $\psi_{\boldsymbol{j}} \in \mathbb{C}^{N^n}$. The Hilbert spaces $L^2(\mathbb{T}^n)$ and $\mathbb{C}^{N^n}$ are isomorphic as shown below

$$\begin{aligned}\langle \phi(\boldsymbol{x}), \psi(\boldsymbol{x})\rangle_{L^2(\mathbb{T}^n)} &= \int_{\mathbb{T}^n} \mathrm{d}\boldsymbol{x}\, \overline{\phi}(\boldsymbol{x})\psi(\boldsymbol{x}) = \sum_{\boldsymbol{j},\boldsymbol{j}'\in\mathbb{Z}_N^n} \overline{\phi}_{\boldsymbol{j}'}\psi_{\boldsymbol{j}} \int_{\mathbb{T}^n} e_{\boldsymbol{j}'}(\boldsymbol{x}) e_{\boldsymbol{j}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &= \sum_{\boldsymbol{j}\in\mathbb{Z}_N^n} \overline{\phi}_{\boldsymbol{j}}\psi_{\boldsymbol{j}} = \langle \phi, \psi\rangle_{\mathbb{C}^{N^n}}.\end{aligned} \tag{3.2.40}$$

We will now solely work with $\mathbb{C}^{N^n}$ and therefore define the Hilbert space in what follows to be $\mathcal{H}_{N^n} := (\mathbb{C}^{N^n}, \langle\cdot,\cdot\rangle_{\mathbb{C}^{N^n}})$.

To obtain a quantisation of symbols on a torus phase space one route is by discretising the configuration space such that the Hilbert space becomes $L^2(\mathbb{Z}_N^n)$ such that $\psi(\boldsymbol{z}) \mapsto \psi(\boldsymbol{j}\frac{\ell_q}{N})$, (this is also seen as "wavefunctions", $\psi_{\boldsymbol{j}}$, supported at points $\boldsymbol{j}\frac{\ell_q}{N}$ in the configuration space). Using (3.2.25), (3.2.26), and (3.2.30) yields

$$\begin{aligned}\left(\mathrm{Op}_\hbar^W[f]\psi\right)(\boldsymbol{z}) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{l},\boldsymbol{z}-\boldsymbol{w}\rangle} \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}} f_{\boldsymbol{n},\boldsymbol{m}} \mathrm{e}^{2\pi i\left(\frac{\langle \boldsymbol{n},\boldsymbol{z}+\boldsymbol{w}\rangle}{2\ell_q} - \frac{\langle \boldsymbol{m},\boldsymbol{l}\rangle}{\ell_p}\right)} \\ &\cdot \psi(\boldsymbol{w})\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{l} \\ &= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^n}\int_{\mathbb{R}^n} f_{\boldsymbol{m},\boldsymbol{n}}\delta\left(\boldsymbol{z}-\boldsymbol{w}-\frac{2\pi\hbar}{\ell_p}\boldsymbol{m}\right)\mathrm{e}^{\frac{\pi i}{\ell_q}\langle \boldsymbol{n},\boldsymbol{z}+\boldsymbol{w}\rangle} \\ &\cdot \psi(\boldsymbol{w})\,\mathrm{d}\boldsymbol{w} \\ &= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^n} f_{\boldsymbol{m},\boldsymbol{n}} \mathrm{e}^{\frac{2\pi i\hbar}{\ell_q}\langle \boldsymbol{n},\boldsymbol{z}\rangle - \frac{2\pi^2 i\hbar}{\ell_q\ell_p}\langle \boldsymbol{n},\boldsymbol{m}\rangle}\psi\left(\boldsymbol{z}-\frac{2\pi\hbar}{\ell_p}\boldsymbol{m}\right) \\ &= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^n} f_{\boldsymbol{n},\boldsymbol{m}} \mathrm{e}^{\frac{2\pi i}{N}\langle \boldsymbol{n},\boldsymbol{j}\rangle - \frac{\pi i}{N}\langle \boldsymbol{n},\boldsymbol{m}\rangle}\psi_{\boldsymbol{j}-\boldsymbol{m}}.\end{aligned} \tag{3.2.41}$$

Defining

$$U\left(\frac{\ell_p \boldsymbol{n}}{N}, \frac{\ell_q \boldsymbol{m}}{N}\right) := T^{\boldsymbol{m},\boldsymbol{n}}, \tag{3.2.42}$$

Weyl quantisation on the torus can thus be seen as replacing the translation operators by its discrete analogue. In each case, we obtain an expression for the Weyl quantisation of a periodic symbol to be

$$\mathrm{Op}_\hbar^W[f] = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^n} f_{\boldsymbol{m},\boldsymbol{n}} T^{\boldsymbol{m},\boldsymbol{n}}, \tag{3.2.43}$$

where the $f_{\boldsymbol{m},\boldsymbol{n}}$ are the Fourier coefficients of f.

We now prove a simple result that shows that if the symbol is real, the corresponding operator is self-adjoint. This shows that Weyl quantisation on the torus phase space yields self-adjoint operators upon quantisation.

Theorem 3.2.1. *The Weyl quantisation of a symbol is self-adjoint if the symbol $f \in \mathscr{S}(\mathbb{R}^{2n})$ is real.*

Proof. From (3.3.29) we define the kernel to be

$$K_f(\boldsymbol{w}, \boldsymbol{z}) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{l}\, \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{z}-\boldsymbol{w}, \boldsymbol{l}\rangle} f\left(\frac{\boldsymbol{z}+\boldsymbol{w}}{2}, \boldsymbol{l}\right) \psi(\boldsymbol{w}) \tag{3.2.44}$$

so we obtain

$$\left(\mathrm{Op}_\hbar^W[f]\psi\right)(\boldsymbol{z}) = \int_{\mathbb{R}^n} K_f(\boldsymbol{w}, \boldsymbol{z})\psi(\boldsymbol{w})\,\mathrm{d}\boldsymbol{w}. \tag{3.2.45}$$

We therefore obtain a self-adjoint operator if

$$K_f^\dagger(\boldsymbol{w}, \boldsymbol{z}) = K_f(\boldsymbol{w}, \boldsymbol{z}). \tag{3.2.46}$$

Taking the adjoint we find

$$K_f^\dagger(\boldsymbol{w}, \boldsymbol{z}) = K_f^*(\boldsymbol{z}, \boldsymbol{w}) = \frac{1}{(2\pi\hbar)^n} \int \mathrm{e}^{i\langle \boldsymbol{z}-\boldsymbol{w}, \boldsymbol{l}\rangle/\hbar} f^*\left(\frac{\boldsymbol{z}+\boldsymbol{w}}{2}, \boldsymbol{l}\right) \mathrm{d}\boldsymbol{l}, \tag{3.2.47}$$

where f^* is the complex conjugate of f. The two operators are therefore equal if $f = f^*$ or if f is real. From proposition 3.1.1, we see that a real symbol gives a self-adjoint operator if $\mathrm{Op}_{\hbar}^{W}[f]$ is bounded. It can be seen that from [Zwo12] theorem 4.21, that quantisation of symbols which belong to Schwartz space, $\mathscr{S}(\mathbb{R}^{2n})$, yield bounded operators. Therefore a quantisation of a real symbol which is an element of Schwartz space (which we will only be considering in this thesis) yields self-adjoint operators. $\square$

Having developed Weyl quantisation on torus phase space which yields self-adjoint operators, we are now in a position to generalise this formulation to include a vector potential. This will be the focus of the next section.

3.3 Magnetic Weyl calculus

The inclusion of a magnetic field to the Weyl calculus was first looked at by [OTSV86, Mü99] and later formulated in a mathematically rigorous way by [MM04]. The case of a compact configuration manifold with a magnetic field has been studied by several authors, [Ono08, Ono01, Lé95, MVK02, CK96, VAG96, Wil84, MAH09]. This section studies Weyl quantisation in the presence of a magnetic field on $\mathbb{R}^{2n}$ and $\mathbb{T}^{2n}$. We follow mainly [MM04] for the mathematical formulation of the magnetic Weyl calculus. In the rest of the thesis we use the convention that $c = e = m = 1$.

3.3.1 Magnetic Weyl quantisation on $\mathbb{R}^{2n}$

The introduction of a magnetic field into quantum mechanics is seen by applying the minimal coupling principle which involves shifting the momentum variables in the Hamiltonian by the corresponding vector potential terms to take into account the effect of the magnetic field. Applying (3.3.29) to a vector $\psi \in L^2(\mathbb{R}^n)$ one could speculate that the correct Weyl quantisation in the presence of a magnetic field is

$$\begin{aligned}
&\left(\mathrm{Op}_\hbar^{\mathcal{A}}[f]\psi\right)(\boldsymbol{z}) \\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\mathrm{d}\boldsymbol{w}\int_{\mathbb{R}^n}\mathrm{d}\boldsymbol{l}\ \mathrm{e}^{i\langle\boldsymbol{z}-\boldsymbol{w},\boldsymbol{l}\rangle/\hbar} f\left(\frac{\boldsymbol{z}+\boldsymbol{w}}{2},\boldsymbol{l}-\boldsymbol{A}\left(\frac{\boldsymbol{z}+\boldsymbol{w}}{2}\right)\right)\psi(\boldsymbol{w}) \\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\mathrm{d}\boldsymbol{w}\int_{\mathbb{R}^n}\mathrm{d}\boldsymbol{l}\ \mathrm{e}^{i\langle\boldsymbol{z}-\boldsymbol{w},\boldsymbol{l}\rangle/\hbar}\mathrm{e}^{i\langle\boldsymbol{z}-\boldsymbol{w},\boldsymbol{A}\left(\frac{\boldsymbol{z}+\boldsymbol{w}}{2}\right)\rangle/\hbar} f\left(\frac{\boldsymbol{z}+\boldsymbol{w}}{2},\boldsymbol{l}\right)\psi(\boldsymbol{w})
\end{aligned} \tag{3.3.1}$$

One realises that this formula is only correct for a limited class of vector potentials (when $\boldsymbol{A}$ is linear vector function) since gauge covariance is otherwise broken (see [MM04]).

The magnetic Weyl calculus can be obtained in a similar manner to the non-magnetic case. We start off with the following operator

$$U^{\boldsymbol{A}}(\boldsymbol{p},\boldsymbol{q}) = \mathrm{e}^{\frac{i}{\hbar}\left(\boldsymbol{p}\cdot\hat{\boldsymbol{Q}}-\boldsymbol{q}\cdot\hat{\boldsymbol{P}}+\boldsymbol{q}\cdot\boldsymbol{A}(\hat{\boldsymbol{Q}})\right)}. \tag{3.3.2}$$

Defining as before

$$g_{\boldsymbol{A}}(\boldsymbol{x},s) = \left(\mathrm{e}^{\frac{i}{\hbar}\left(s\boldsymbol{p}\cdot\hat{\boldsymbol{Q}}-s\boldsymbol{q}\cdot\hat{\boldsymbol{P}}+s\boldsymbol{q}\cdot\boldsymbol{A}(\hat{\boldsymbol{Q}})\right)}f\right)(\boldsymbol{x}), \tag{3.3.3}$$

we find

$$\frac{\partial g_{\boldsymbol{A}}}{\partial s} + \langle\boldsymbol{q},\partial_{\boldsymbol{x}}g_{\boldsymbol{A}}\rangle = \frac{i}{\hbar}\langle\boldsymbol{p},\boldsymbol{x}\rangle g_{\boldsymbol{A}} + \frac{i}{\hbar}\langle\boldsymbol{q},\boldsymbol{A}(\boldsymbol{x})\rangle g_{\boldsymbol{A}} \tag{3.3.4}$$

and letting $G(s) = g_{\boldsymbol{A}}(\boldsymbol{x}+s\boldsymbol{q},s)$, the above reduces to

$$\frac{\mathrm{d}G(s)}{\mathrm{d}s} = \left(\frac{i}{\hbar}\langle\boldsymbol{p},\boldsymbol{x}+s\boldsymbol{q}\rangle + \frac{i}{\hbar}\langle\boldsymbol{q},\boldsymbol{A}(\boldsymbol{x}+s\boldsymbol{q})\rangle\right)G(s). \tag{3.3.5}$$

Solving yields

$$G(s) = \mathrm{e}^{\frac{i}{\hbar}s\langle\boldsymbol{p},\boldsymbol{x}\rangle+\frac{i}{2\hbar}s^2\langle\boldsymbol{p},\boldsymbol{q}\rangle+\frac{i}{\hbar}\int_0^s\langle\boldsymbol{q},\boldsymbol{A}(\boldsymbol{x}+s'\boldsymbol{q})\rangle\mathrm{d}s'}f(\boldsymbol{x}) \tag{3.3.6}$$

where again, $G(0) = f(\boldsymbol{x})$. Letting $s = 1$, $\boldsymbol{x}\mapsto\boldsymbol{x}-\boldsymbol{q}$ and simplifying integration variable in the vector potential term we obtain

$$\left(U^{\mathcal{A}}(\boldsymbol{p},\boldsymbol{q})f\right)(\boldsymbol{x}) = \mathrm{e}^{\frac{i}{\hbar}\langle\boldsymbol{p},\boldsymbol{x}\rangle-\frac{i}{2\hbar}\langle\boldsymbol{p},\boldsymbol{q}\rangle+\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{q}])}f(\boldsymbol{x}-\boldsymbol{q}), \tag{3.3.7}$$

where

$$\int_0^1 \langle \boldsymbol{q}, \boldsymbol{A}(\boldsymbol{x}-\boldsymbol{q}s)\rangle \mathrm{d}s := \Gamma^{\mathcal{A}}([\boldsymbol{x}, \boldsymbol{x}-\boldsymbol{q}]), \tag{3.3.8}$$

and

$$\Gamma^{\mathcal{C}}(\gamma) := \int_\gamma \mathcal{C} = \int_\gamma \langle \boldsymbol{C}(\boldsymbol{x}), \mathrm{d}\boldsymbol{x}\rangle, \tag{3.3.9}$$

$\mathrm{d}\boldsymbol{x}$ is a vector of differentials with $\boldsymbol{C}(\boldsymbol{x}) = (C_1(\boldsymbol{x}), C_2(\boldsymbol{x}), ..., C_n(\boldsymbol{x}))$. The above has the interpretation of the circulation of a 1-form $\mathcal{C}$ along a line segment γ. In this thesis the common circulations are 1-forms along line segments $\gamma = [\boldsymbol{x}, \boldsymbol{y}]$ and 2-forms through parallelograms with vertices $\{\boldsymbol{z}, \boldsymbol{z}+\boldsymbol{x}, \boldsymbol{z}+\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{z}+\boldsymbol{y}\}$.

Weyl quantisation in the presence of a magnetic field can proceed in a way similar to (3.2.22) where we expand the operator with a symplectic Fourier transform. Proceeding as before we obtain

$$\left(\mathrm{Op}_\hbar^{\mathcal{A}}[f]\psi\right)(\boldsymbol{z}) = \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{\eta}\, \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{y},\boldsymbol{p}\rangle - \langle \boldsymbol{l},\boldsymbol{q}\rangle + \frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{z},\boldsymbol{z}-\boldsymbol{q}])} f(\boldsymbol{y}, \boldsymbol{l}) \tag{3.3.10}$$

$$\cdot\, \mathrm{e}^{\frac{i}{\hbar}(\langle \boldsymbol{p},\boldsymbol{z}\rangle - \frac{1}{2}\langle \boldsymbol{p},\boldsymbol{q}\rangle)}\psi(\boldsymbol{z}-\boldsymbol{q})$$

$$= \frac{1}{(2\pi\hbar)^{n}} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{\eta}'\delta\left(\boldsymbol{z}-\boldsymbol{y}-\frac{1}{2}\boldsymbol{q}\right) \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{q},\boldsymbol{l}\rangle + \frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{z},\boldsymbol{z}-\boldsymbol{q}])} \tag{3.3.11}$$

$$\cdot\, f(\boldsymbol{y}, \boldsymbol{l})\psi(\boldsymbol{z}-\boldsymbol{q})$$

$$= \frac{1}{(2\pi\hbar)^{n}} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{q} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{l}\, \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{q},\boldsymbol{l}\rangle + \frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{z},\boldsymbol{z}-\boldsymbol{q}])}$$

$$\cdot\, f\left(\boldsymbol{z}-\frac{1}{2}\boldsymbol{q}, \boldsymbol{l}\right)\psi(\boldsymbol{z}-\boldsymbol{q}), \tag{3.3.12}$$

where $\int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{\eta}' = \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{q} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{y} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{l}$. Letting $\boldsymbol{w} = \boldsymbol{z}-\boldsymbol{q}$ yields

$$\left(\mathrm{Op}_\hbar^{\mathcal{A}}[f]\psi\right)(\boldsymbol{z}) = \frac{1}{(2\pi\hbar)^{n}} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{\chi}\, \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{z}-\boldsymbol{w},\boldsymbol{l}\rangle + \frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{z},\boldsymbol{w}])} f\left(\frac{\boldsymbol{z}+\boldsymbol{w}}{2}, \boldsymbol{l}\right)\psi(\boldsymbol{w}), \tag{3.3.13}$$

where

$$\Gamma^{\mathcal{A}}([\boldsymbol{z}, \boldsymbol{w}]) = \int_0^1 \langle \boldsymbol{z}-\boldsymbol{w}, \boldsymbol{A}(\boldsymbol{z}+s(\boldsymbol{w}-\boldsymbol{z}))\rangle\, \mathrm{d}s \tag{3.3.14}$$

and

$$\int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{\chi} = \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{w} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{l}.$$

As mentioned at the start of this section, the usual minimal coupling principal in the non-magnetic Weyl calculus yields a quantisation scheme which does not obey gauge covariance. We show that the magnetic Weyl calculus just obtained does yield a gauge covariant quantisation scheme.

Under a gauge transformation $\mathcal{A} \mapsto \mathcal{A} + \mathrm{d}\chi$, (3.3.7) becomes

$$\begin{aligned}\left(U^{\mathcal{A}+\mathrm{d}\chi}(\boldsymbol{p},\boldsymbol{q})f\right)(\boldsymbol{x}) &= \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{p},\boldsymbol{x}\rangle - \frac{i}{2\hbar}\langle \boldsymbol{p},\boldsymbol{q}\rangle + \frac{i}{\hbar}\int_{[\boldsymbol{x},\boldsymbol{x}-\boldsymbol{q}]}(\mathcal{A}+\mathrm{d}\chi)} f(\boldsymbol{x}-\boldsymbol{q}) \\ &= \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{p},\boldsymbol{x}\rangle - \frac{i}{2\hbar}\langle \boldsymbol{p},\boldsymbol{q}\rangle + \frac{i}{\hbar}\int_{[\boldsymbol{x},\boldsymbol{x}-\boldsymbol{q}]}\mathcal{A} + \frac{i}{\hbar}\chi(\boldsymbol{x}-\boldsymbol{q}) - \frac{i}{\hbar}\chi(\boldsymbol{x})} f(\boldsymbol{x}-\boldsymbol{q}) \\ &= \left(\mathrm{e}^{\frac{i}{\hbar}\chi(\hat{\boldsymbol{Q}})} U^{\mathcal{A}}(\boldsymbol{p},\boldsymbol{q}) \mathrm{e}^{-\frac{i}{\hbar}\chi(\hat{\boldsymbol{Q}})} f\right)(\boldsymbol{x}),\end{aligned} \tag{3.3.15}$$

which is unitarily equivalent to $U^{\mathcal{A}}(\boldsymbol{p},\boldsymbol{q})$ and thus the magnetic Weyl calculus is gauge covariant.

3.3.2 Magnetic Weyl quantisation on $\mathbb{T}^{2n}$

In the previous section we saw that the restriction to the torus allowed one to obtain a quantisation condition which restricted the allowed values $\hbar$ could take. We now apply the same logic to the magnetic translation operators by first looking at general translations in independent directions then, applying this to the specific cases of the Landau vector potential $\boldsymbol{A}(\boldsymbol{x}) = (-bx^2, 0)$ and a non-homogeneous magnetic field obtained by the taking terms from a Fourier series expansion of the Henon-Heiles potential.

Using (3.3.7) repeatedly, we find

$$\begin{aligned}\left(U^{\mathcal{A}}(\boldsymbol{p},\boldsymbol{q})U^{\mathcal{A}}(\boldsymbol{l},\boldsymbol{y})f\right)(\boldsymbol{x}) &= \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{l},\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\rangle + \frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{y}])} \left(U^{\boldsymbol{A}}(\boldsymbol{p},\boldsymbol{q})f\right)(\boldsymbol{x}) \\ &= \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{l},\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\rangle + \frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{y}])} \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{p},\boldsymbol{x}-\boldsymbol{y}-\frac{1}{2}\boldsymbol{q}\rangle} \\ &\cdot \mathrm{e}^{\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}])} f(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}).\end{aligned} \tag{3.3.16}$$

Computing

$$\left(U^{\mathcal{A}}(\boldsymbol{l}+\boldsymbol{p},\boldsymbol{y}+\boldsymbol{q})f\right)(\boldsymbol{x}) = \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{l}+\boldsymbol{p},\boldsymbol{x}-\frac{1}{2}(\boldsymbol{y}+\boldsymbol{q})\rangle+\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}])} f(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}), \tag{3.3.17}$$

we insert "1"s into (3.3.16) and compare with (3.3.17) to obtain

$$\begin{aligned}\left(U^{\mathcal{A}}(\boldsymbol{p},\boldsymbol{q})U^{\mathcal{A}}(\boldsymbol{l},\boldsymbol{y})f\right)(\boldsymbol{x}) &= \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{l},\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\rangle+\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{y}])}\mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{p},\boldsymbol{x}-\boldsymbol{y}-\frac{1}{2}\boldsymbol{q}\rangle}\\ &\quad\cdot \mathrm{e}^{\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}])} f(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q})\\ &= \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{l},\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\rangle+\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{y}])}\mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{p},\boldsymbol{x}-\boldsymbol{y}-\frac{1}{2}\boldsymbol{q}\rangle+\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}])}\\ &\quad\cdot \mathrm{e}^{-\frac{i}{2\hbar}\langle \boldsymbol{l},\boldsymbol{q}\rangle+\frac{i}{2\hbar}\langle \boldsymbol{l},\boldsymbol{q}\rangle+\frac{i}{2\hbar}\langle \boldsymbol{p},\boldsymbol{y}\rangle-\frac{i}{2\hbar}\langle \boldsymbol{p},\boldsymbol{y}\rangle+\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}])}\\ &\quad\cdot \mathrm{e}^{-\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}])} f(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q})\\ &= \mathrm{e}^{-\frac{i}{2\hbar}\sigma((\boldsymbol{q},\boldsymbol{p}),(\boldsymbol{y},\boldsymbol{l}))}\omega^{\mathcal{B}}(\boldsymbol{x};\boldsymbol{y},\boldsymbol{q})\left(U^{\mathcal{A}}(\boldsymbol{l}+\boldsymbol{p},\boldsymbol{y}+\boldsymbol{q})f\right)(\boldsymbol{x})\end{aligned} \tag{3.3.18}$$

where

$$\omega^{\mathcal{B}}(\boldsymbol{x};\boldsymbol{y},\boldsymbol{q}) := \mathrm{e}^{\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{y}])+\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}])-\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}])}. \tag{3.3.19}$$

We therefore find

$$\begin{aligned}U^{\mathcal{A}}(\boldsymbol{p},\boldsymbol{q})U^{\mathcal{A}}(\boldsymbol{l},\boldsymbol{y}) = \mathrm{e}^{-\frac{i}{\hbar}\sigma((\boldsymbol{q},\boldsymbol{p}),(\boldsymbol{y},\boldsymbol{l}))}&\omega^{\mathcal{B}}(\hat{\boldsymbol{Q}};\boldsymbol{y},\boldsymbol{q})\left[\omega^{\mathcal{B}}(\hat{\boldsymbol{Q}};\boldsymbol{q},\boldsymbol{y})\right]^{-1}\\ &\cdot U^{\mathcal{A}}(\boldsymbol{l},\boldsymbol{y})U^{\mathcal{A}}(\boldsymbol{p},\boldsymbol{q}).\end{aligned} \tag{3.3.20}$$

On the torus phase space we again condition the translation operators to commute; this restriction yields (omitting the vector)

$$\mathrm{e}^{-\frac{i}{\hbar}\sigma((\boldsymbol{q},\boldsymbol{p}),(\boldsymbol{y},\boldsymbol{l}))}\omega^{\mathcal{B}}(\boldsymbol{x};\boldsymbol{y},\boldsymbol{q})\left[\omega^{\mathcal{B}}(\boldsymbol{x};\boldsymbol{q},\boldsymbol{y})\right]^{-1} = 1 \tag{3.3.21}$$

where

$$\begin{aligned}\omega^{\mathcal{B}}(\boldsymbol{x};\boldsymbol{y},\boldsymbol{q})\left[\omega^{\mathcal{B}}(\boldsymbol{x};\boldsymbol{q},\boldsymbol{y})\right]^{-1} &= \mathrm{e}^{\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{y}])+\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}])-\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{y}])}\\ &\quad\cdot \mathrm{e}^{-\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{q}])-\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x}-\boldsymbol{q},\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{y}])+\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{y}])}\\ &= \mathrm{e}^{\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{y}])+\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}])}\\ &\quad\cdot \mathrm{e}^{-\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x},\boldsymbol{x}-\boldsymbol{q}])-\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{x}-\boldsymbol{q},\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{y}])}\\ &= \mathrm{e}^{\frac{i}{\hbar}\Xi^{\mathcal{B}}(\boldsymbol{x};\boldsymbol{q},\boldsymbol{y})}.\end{aligned} \tag{3.3.22}$$

where

$$\begin{aligned}\Xi^{\mathcal{B}}(\boldsymbol{x};\boldsymbol{q},\boldsymbol{y}) &= \int_0^1 \langle \boldsymbol{y}, \boldsymbol{A}(\boldsymbol{x}-s\boldsymbol{y})\rangle \mathrm{d}s + \int_0^1 \langle \boldsymbol{q}, \boldsymbol{A}(\boldsymbol{x}-\boldsymbol{y}-s\boldsymbol{q})\rangle \mathrm{d}s \\ &- \int_0^1 \langle \boldsymbol{q}, \boldsymbol{A}(\boldsymbol{x}-s\boldsymbol{q})\rangle \mathrm{d}s - \int_0^1 \langle \boldsymbol{y}, \boldsymbol{A}(\boldsymbol{x}-\boldsymbol{q}-s\boldsymbol{y})\rangle \mathrm{d}s \\ &:= \int_{\{\boldsymbol{x},\boldsymbol{x}-\boldsymbol{q},\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q},\boldsymbol{x}-\boldsymbol{y}\}} \mathcal{B} \end{aligned} \tag{3.3.23}$$

has the interpretation of the flux through the parallelogram with vertices $\{\boldsymbol{x}, \boldsymbol{x}-\boldsymbol{q}, \boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}, \boldsymbol{x}-\boldsymbol{y}\}$.

We now check the commutability requirement for the example of the Landau gauge. The condition for the magnetic translation operators to commute is the following

$$-\sigma((\boldsymbol{q},\boldsymbol{p}),(\boldsymbol{y},\boldsymbol{l})) + \Xi^{\mathcal{B}}(\boldsymbol{x};\boldsymbol{q},\boldsymbol{y}) = 2\pi\hbar m, \text{ where } m \in \mathbb{Z}. \tag{3.3.24}$$

With the vector potential $\boldsymbol{A}(\boldsymbol{x}) = (-bx^2, 0)$, where $\mathcal{B} = \mathrm{d}\mathcal{A}$, along with (3.3.22) and (3.3.14) we obtain

$$\langle \boldsymbol{l}, \boldsymbol{q}\rangle - \langle \boldsymbol{p}, \boldsymbol{y}\rangle + b(q^1y^2 - q^2y^1) = 2\pi\hbar m. \tag{3.3.25}$$

By considering different translations around the torus phase space we obtain two independent conditions:

$$\ell_q\ell_p = 2\pi\hbar N, \text{ where } N \in \mathbb{Z}; \tag{3.3.26a}$$

$$b\ell_q^2 = 2\pi\hbar M, \text{ where } M \in \mathbb{Z}. \tag{3.3.26b}$$

We find the original phase space quantisation condition and an additional condition which is the flux through the configuration space torus. We have already seen that when the phase space is restricted to that of a torus, the condition $\ell_q\ell_p = 2\pi\hbar N$ corresponds a phase space quantisation condition where N is the dimension of the Hilbert space. The flux quantisation condition implies that the flux through the configuration space is to be an integer multiple of Planck's constant, h. In addition, it is seen from (3.3.26b) that in the semiclassical limit ($\hbar \to 0$) an equivalent limit is obtained ($M \to \infty$) similar to that of the phase space quantisation condition.

We now discuss the commutation of the magnetic translation operators for the case of a non-homogeneous magnetic field. The vector potential resides only on the configuration space which in our case is $\mathbb{T}^2 := \mathbb{R}^2/\ell_q\mathbb{Z}^2$, thus, functions on $\mathbb{T}^2$ are periodic with respect to the fundamental lengths, ℓ_q of the torus. The vector potential thus has the following form

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{x}) &= \left(-bx^2 + \sum_{\boldsymbol{l}\in\mathbb{Z}^2/\{\boldsymbol{0}\}} \alpha^1_{\boldsymbol{l}} \mathrm{e}^{\frac{2\pi i}{\ell_q}\langle \boldsymbol{l},\boldsymbol{x}\rangle}, \sum_{\boldsymbol{s}\in\mathbb{Z}^2/\{\boldsymbol{0}\}} \alpha^2_{\boldsymbol{s}} \mathrm{e}^{\frac{2\pi i}{\ell_q}\langle \boldsymbol{s},\boldsymbol{x}\rangle}\right)\\ &= (-bx^2, 0) + \boldsymbol{A}'(\boldsymbol{x}) \end{aligned} \tag{3.3.27}$$

where $\boldsymbol{A}'(\boldsymbol{x})$ is the periodic contribution to the vector potential $\boldsymbol{A}(\boldsymbol{x})$. Since $\boldsymbol{A}'(\boldsymbol{x}+\boldsymbol{\alpha}\ell_q) = \boldsymbol{A}'(\boldsymbol{x})$ where $\boldsymbol{\alpha} \in \mathbb{Z}^2$, and noting that we are interested in translations over the length of the torus configuration space, i.e., $\boldsymbol{q} = \ell_q\boldsymbol{n}$ and $\boldsymbol{y} = \ell_q\boldsymbol{r}$ where $\boldsymbol{n}, \boldsymbol{r} \in \mathbb{Z}^2$, we find from (3.3.23)

$$\begin{aligned}\Xi^{\mathcal{B}}(\boldsymbol{x}; \ell_q\boldsymbol{n}, \ell_q\boldsymbol{r}) &= b\ell_q^2(n_1r_2 - n_2r_1)\\ &\quad + \int_0^1 \langle \ell_q\boldsymbol{r}, \boldsymbol{A}'(\boldsymbol{x} - s\ell_q\boldsymbol{r})\rangle\,\mathrm{d}s - \int_0^1 \langle \ell_q\boldsymbol{n}, \boldsymbol{A}'(\boldsymbol{x} - s\ell_q\boldsymbol{n})\rangle\,\mathrm{d}s\\ &\quad + \int_0^1 \langle \ell_q\boldsymbol{n}, \boldsymbol{A}'(\boldsymbol{x} - s\ell_q\boldsymbol{n})\rangle\,\mathrm{d}s - \int_0^1 \langle \ell_q\boldsymbol{r}, \boldsymbol{A}'(\boldsymbol{x} - s\ell_q\boldsymbol{r})\rangle\,\mathrm{d}s\\ &= b\ell_q^2(n_1r_2 - n_2r_1), \end{aligned} \tag{3.3.28}$$

therefore for a general vector potential in $\mathbb{T}^2$ the flux quantisation has the form $b\ell_q^2 = 2\pi\hbar M$ for some $M \in \mathbb{Z}$. Since the magnetic translation commutation relation, (3.3.20) holds for $\mathbb{R}^n$ we can generalise (3.3.24) to $\mathbb{T}^n$ and again conclude with the same argument as for $\mathbb{T}^2$, that a periodic vector potential will only contribute $b\ell_q^2$ to the flux quantisation condition. Therefore the quantisation conditions (3.3.26a) and (3.3.26b) hold for general vector potentials on configuration space $\mathbb{T}^n$ and phase space $\mathbb{T}^{2n}$.

Proceeding as we did for the non-magnetic Weyl calculus, we discretise the configuration space to obtain states which are elements of $\mathbb{C}^{N^n}$. Quantising

(3.2.26) using (3.3.13) yields

$$
\begin{aligned}
\left(\mathrm{Op}_\hbar^{\mathcal{A}}[f]\psi\right)(\boldsymbol{z}) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{\chi}\, \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{l},\boldsymbol{z}-\boldsymbol{w}\rangle+\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{z},\boldsymbol{w}])} \\
&\cdot \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}} f_{\boldsymbol{n},\boldsymbol{m}} \mathrm{e}^{2\pi i\left(\frac{\langle \boldsymbol{n},\boldsymbol{z}+\boldsymbol{w}\rangle}{2\ell_q}-\frac{\langle \boldsymbol{m},\boldsymbol{l}\rangle}{\ell_p}\right)}\psi(\boldsymbol{w}) \\
&= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}} f_{\boldsymbol{n},\boldsymbol{m}} \int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{w}\, \delta\left(\boldsymbol{z}-\boldsymbol{w}-\frac{2\pi\hbar}{\ell_p}\boldsymbol{m}\right) \\
&\cdot \mathrm{e}^{\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{z},\boldsymbol{w}])+\frac{\pi i}{\ell_q}(\boldsymbol{z}+\boldsymbol{w})}\psi(\boldsymbol{w}) \\
&= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}} f_{\boldsymbol{n},\boldsymbol{m}} \mathrm{e}^{\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{z},\boldsymbol{z}-\frac{2\pi\hbar}{\ell_p}\boldsymbol{m}])+\frac{\pi i}{\ell_q}\langle \boldsymbol{n},2\boldsymbol{z}+\frac{2\pi\hbar}{\ell_p}\boldsymbol{m}\rangle}\psi\left(\boldsymbol{z}-\frac{2\pi\hbar}{\ell_p}\right) \\
&= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}} f_{\boldsymbol{n},\boldsymbol{m}} \mathrm{e}^{\frac{i}{\hbar}\gamma^{\mathcal{A}}([\boldsymbol{j},\boldsymbol{j}-\boldsymbol{m}])+\frac{2\pi i}{N}\langle \boldsymbol{n},\boldsymbol{j}\rangle-\frac{\pi i}{N}\langle \boldsymbol{n},\boldsymbol{m}\rangle}\psi_{\boldsymbol{j}-\boldsymbol{m}}, \qquad (3.3.29)
\end{aligned}
$$

where

$$
\gamma^{\mathcal{A}}([\boldsymbol{j},\boldsymbol{j}-\boldsymbol{m}]) := \frac{\ell_q}{N}\int_0^1 \langle \boldsymbol{m}, \boldsymbol{A}\left(\frac{\ell_q}{N}(\boldsymbol{j}-s\boldsymbol{m})\right)\rangle \mathrm{d}s. \qquad (3.3.30)
$$

We can therefore define the magnetic translations as

$$
\left(T_{\mathcal{A}}^{\boldsymbol{m},\boldsymbol{n}}\psi\right)_{\boldsymbol{j}} := \mathrm{e}^{\frac{i}{\hbar}\gamma^{\mathcal{A}}([\boldsymbol{j},\boldsymbol{j}-\boldsymbol{m}])+\frac{2\pi i}{N}\langle \boldsymbol{n},\boldsymbol{j}\rangle-\frac{\pi i}{N}\langle \boldsymbol{n},\boldsymbol{m}\rangle}\psi_{\boldsymbol{j}-\boldsymbol{m}}. \qquad (3.3.31)
$$

Using (3.3.29) we obtain an operator expansion in terms of magnetic translation operators

$$
\mathrm{Op}_\hbar^{\mathcal{A}}[f] = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}} f_{\boldsymbol{n},\boldsymbol{m}} T_{\mathcal{A}}^{\boldsymbol{m},\boldsymbol{n}}. \qquad (3.3.32)
$$

We now show the trivial case of when the symbol f is real, the magnetic Weyl quantisation of f yields a self-adjoint operator.

Theorem 3.3.1. *The magnetic Weyl quantisation of a symbol is self-adjoint if the symbol f is real.*

Proof. We start by noting that the operator (3.3.35) can be written as

$$
\begin{aligned}
\left(\mathrm{Op}_\hbar^{\mathcal{A}}[f]\psi\right)(\boldsymbol{z}) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} \mathrm{d}\boldsymbol{\chi}\, \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{l},\boldsymbol{z}-\boldsymbol{w}\rangle+\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{z},\boldsymbol{w}])} f\left(\frac{\boldsymbol{z}+\boldsymbol{w}}{2},\boldsymbol{l}\right)\psi(\boldsymbol{w}) \\
&= \int_{\mathbb{R}^n} K_f^{\mathcal{A}}(\boldsymbol{z},\boldsymbol{w})\psi(\boldsymbol{w})\,\mathrm{d}\boldsymbol{w}
\end{aligned}
\qquad (3.3.33)
$$

where

$$K_f^{\mathcal{A}}(\boldsymbol{z},\boldsymbol{x}) := \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{l},\boldsymbol{z}-\boldsymbol{w}\rangle}\mathrm{e}^{\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{z},\boldsymbol{w}])} f\left(\frac{\boldsymbol{z}+\boldsymbol{w}}{2},\boldsymbol{l}\right)\,\mathrm{d}\boldsymbol{l}. \tag{3.3.34}$$

Taking the adjoint of (3.3.34) we find

$$\begin{aligned}
\left(\mathrm{Op}_\hbar^{\mathcal{A}}[f]^\dagger\psi\right)(\boldsymbol{z}) &= \int_{\mathbb{R}^n}\left(K_f^{\mathcal{A}}(\boldsymbol{z},\boldsymbol{w})\right)^\dagger \psi(\boldsymbol{w})\,\mathrm{d}\boldsymbol{w},\\
&= \int_{\mathbb{R}^n} K_f^{\mathcal{A}}(\boldsymbol{w},\boldsymbol{z})^*\psi(\boldsymbol{w})\,\mathrm{d}\boldsymbol{w}\\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}}\int_{\mathbb{R}^n} \mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{l},\boldsymbol{z}-\boldsymbol{w}\rangle}\mathrm{e}^{\frac{i}{\hbar}\Gamma^{\mathcal{A}}([\boldsymbol{z},\boldsymbol{w}])}\cdot\\
&\quad\cdot f^*\left(\frac{\boldsymbol{x}+\boldsymbol{z}}{2},\boldsymbol{l}\right)\psi(\boldsymbol{w})\,\mathrm{d}\boldsymbol{l}\,\mathrm{d}\boldsymbol{w},
\end{aligned} \tag{3.3.35}$$

where we have used $\Gamma^{\mathcal{A}}([\boldsymbol{w},\boldsymbol{z}]) = -\Gamma^{\mathcal{A}}([\boldsymbol{z},\boldsymbol{w}])$. We thus see from (3.3.35) that the operator is self-adjoint if $f = f^*$ and if $\mathrm{Op}_\hbar^{\mathcal{A}}[f]$ is bounded which can again be shown using theorem 4.21 from [Zwo12]. $\square$

In this chapter we have shown how to obtain the quantisation of a symbol in both the non-magnetic and magnetic Weyl calculus in $\mathbb{R}^n$ and $\mathbb{T}^{2n}$ and obtained a self-adjoint operator if the symbol is real. We have re-derived the Weyl-calculus on the torus phase space by a way that is equivalent to the group theoretic way of the Heisenberg group, but is simply the quantisation of a periodic symbol acting in $\mathcal{H}_{N^n}$, defined in the previous section as $\mathbb{C}^{N^n}$ with the usual inner product on $\mathbb{C}^{N^n}$. We then generalised the Weyl calculus to include a magnetic field in both $\mathbb{R}^n$ and $\mathbb{T}^{2n}$. This will allow one to study, for $n \geq 2$, the quantum kinematics and dynamics on $\mathbb{T}^{2n}$ in the presence of a magnetic field in a gauge covariant formulation.

Chapter 4

Spectral Analysis of the Laplacian

4.1 Laplacian

In this chapter we obtain an expression for the discrete Laplacian by considering translations in the configuration space. With this, we develop the corresponding classical Hamiltonian and the Hamiltonian matrix for the case of non-magnetic field and the Landau gauge. It is then shown that for integer valued magnetic field strengths the minimal coupling principle can be used for the case of the Landau gauge and the classical Hamiltonian can be obtained. We then use the infinite-momentum limit to obtain a comparison to that of the charged particle in the presence of a magnetic field on the configuration space torus, $\mathbb{T}^2 = \mathbb{R}^2/(\ell_q\mathbb{Z} \oplus \ell_q\mathbb{Z})$, [Ono01]. An algorithm is then obtained which numerically computes the spectrum of the Landau Hamiltonian. To check the accuracy of this algorithm we compare the analytic spectra (using the (Einstein-Brillouin-Keller) EBK model) of the non-magnetic Laplacian to the numerically obtained spectra for the case of zero magnetic field. We then use the algorithm to obtain the spectra of the non-homogeneous magnetic field and compute a selection of spectral statistics for the non-magnetic, the Landau gauge and the non-homogeneous cases.

4.2 Background

Before we proceed we will define the lattice and the Laplacian over the lattice which will allow us to obtain the from the non-magnetic Hamiltonian, an expression for the Weyl quantisation of the Laplacian. We will also discuss the EBK model and provide a simple example to illustrate the method. The definitions used in this section are taken from [EdF10] and statement of the EBK model from [MVB76]. Note also, we revert back to the phase space coordinates $(\boldsymbol{q}, \boldsymbol{p})$ rather than $(\boldsymbol{x}, \boldsymbol{\xi})$ which we used to avoid confusion with the translations by vectors $\boldsymbol{q}$ and $\boldsymbol{p}$.

To define the discrete Laplacian over the configuration space $\mathbb{T}^n = \mathbb{R}^n/\ell_q\mathbb{Z}^n$, the Laplacian is restricted by defining a lattice on which it acts and by convention, letting the Laplacian vanish outside the lattice. The configuration space torus is defined as

$$\boldsymbol{\Gamma} = \mathbb{R}^n/(\ell_q\mathbb{Z} \oplus \cdots \oplus \ell_q\mathbb{Z}) := \{\boldsymbol{q} \in \mathbb{R}^n : 0 \leq q^j < \ell_q \text{ for } j = 1, ..., n\}. \tag{4.2.1}$$

Definition 8. *The discrete Laplacian on $\boldsymbol{\Gamma}$, is the difference operator $-\hbar^2\Delta : \mathbb{C}^{\boldsymbol{\Gamma}} \to \mathbb{C}^{\boldsymbol{\Gamma}}$ (where $\mathbb{C}^{\boldsymbol{\Gamma}}$ is defined as the space of $\psi(\boldsymbol{q}) : \boldsymbol{\Gamma} \to \mathbb{C}$) given by*

$$-\hbar^2\Delta\psi(\boldsymbol{q}) := \frac{\hbar^2}{a^2}\sum_{k=1}^{n}\left(2\psi(\boldsymbol{q}) - \psi(\boldsymbol{q} + a\boldsymbol{e}_k) - \psi(\boldsymbol{q} - a\boldsymbol{e}_k)\right)$$

where $\boldsymbol{e}_k$ is the unit vector in the k-th direction. This can be shown with two applications of the first derivative on the lattice with spacing a

$$\left(\partial_{q^1} f\right)(q^1, q^2) := \frac{f(q^1 + a, q^2) - f(q^1, q^2)}{a}, \tag{4.2.2}$$

and the same again for the second component. Therefore, applying definition 8 to $\psi(\boldsymbol{q}) = \psi(\frac{\ell_q \boldsymbol{l}}{N}) = \psi_{\boldsymbol{l}}$ with $a = \ell_q/N$, which discretises the fundamental length of the torus into N points, making $\boldsymbol{\Gamma}$, into a lattice, the discrete Laplacian on $\mathbb{T}^2$ is thus

$$\left(-\hbar^2\Delta\psi\right)_{l_1,l_2} := -\frac{\hbar^2 N^2}{\ell_q^2}\left(\psi_{l_1+1,l_2} + \psi_{l_1-1,l_2} + \psi_{l_1,l_2+1} + \psi_{l_1,l_2-1} - 4\psi_{l_1,l_2}\right) \tag{4.2.3}$$

where $(\psi_{\boldsymbol{l}}) \in \mathbb{C}^{N^2}$ and $\psi_{l^1+N,l^2+N} = \psi_{\boldsymbol{l}}$.

The eigenvalues for quantisations of Liouville integrable systems can be found via the action-angle formulation. Semiclassically, the eigenvalues of a Hamiltonian expressed in action-angle variables can be found by imposing a lattice structure in $\boldsymbol{J}$ space by

$$\boldsymbol{J} = \left(\boldsymbol{m} + \frac{1}{4}\boldsymbol{\alpha}\right)\hbar = \hbar\left(m_1 + \frac{1}{4}\alpha_1, \cdots, m_n + \frac{1}{4}\alpha_n\right)^T, \tag{4.2.4}$$

where $m_i \in \mathbb{Z}_+ := \{0, 1, 2, \cdots\}$ and α_i equals the number of caustics the family $\boldsymbol{J}$ encounters during a cycle of motion corresponding to the period, ω^i. The energy levels are thus found to be the lattice points which touch the energy contour, E. The quantum energies are thus found to be of the form

$$E_{m_1,\ldots,m_n} = H\left(\left(\boldsymbol{m} + \frac{1}{4}\boldsymbol{\alpha}\right)\hbar\right). \tag{4.2.5}$$

As an aside we now find the spectrum of the free particle without a magnetic field on the configuration space torus, $\mathbb{T}^2 = \mathbb{R}^2/(L_1\mathbb{Z} \oplus L_2\mathbb{Z})$. The Hamiltonian of such a system has the following form

$$H(\boldsymbol{p}) = \frac{\langle \boldsymbol{p}, \boldsymbol{p}\rangle_{\mathbb{R}^n}}{2m} = E. \tag{4.2.6}$$

The Hamiltonian above is separable and the two coordinates, q^1 and q^2 are cyclic and therefore the corresponding momentums are constants, $\boldsymbol{\alpha}$. The action variables are found as

$$J_1 = \frac{1}{2\pi}\oint_{\mathcal{C}_1} p_1\,\mathrm{d}q^1 = \frac{1}{2\pi}\int_0^{L_1} \alpha_1\,\mathrm{d}q^1 = \frac{\alpha_1 L_1}{2\pi}, \tag{4.2.7a}$$

$$J_2 = \frac{1}{2\pi}\oint_{\mathcal{C}_2} p_2\,\mathrm{d}q^2 = \frac{1}{2\pi}\int_0^{L_2} \alpha_2\,\mathrm{d}q^2 = \frac{\alpha_2 L_2}{2\pi}. \tag{4.2.7b}$$

Therefore

$$H(\boldsymbol{J}) = \frac{2\pi^2}{m}\left(\frac{J_1^2}{L_1^2} + \frac{J_2^2}{L_2^2}\right). \tag{4.2.8}$$

To obtain a canonical transformation from $(\boldsymbol{p}, \boldsymbol{q}) \to (\boldsymbol{J}, \boldsymbol{\theta})$ the two-form must be invariant,

$$\omega^2 = \sum_{i=1}^{2} \mathrm{d}p_i \wedge \mathrm{d}q^i = \sum_{i=1}^{2} \mathrm{d}J_i \wedge \mathrm{d}\theta^i. \tag{4.2.9}$$

Using (4.2.7a) and (4.2.7b) we find

$$\omega^2 = \sum_{i=1}^{2} \frac{2\pi}{L_i} \mathrm{d}J_i \wedge \mathrm{d}q^i. \tag{4.2.10}$$

Thus, to be a canonical transformation we therefore have $\theta^i = L_i q^i/2\pi$. Using (4.2.5), (4.2.8) and the fact that no caustics occur for straight line trajectories, $\alpha_i = 0$ for $i = 1, 2$ (not to be confused with the constant momentum $\boldsymbol{\alpha}$ above), the EBK energy levels are therefore

$$E_{m_1,m_2} = \frac{2\pi^2}{m}\left(\frac{m_1^2\hbar^2}{L_1} + \frac{m_2^2\hbar^2}{L_2}\right), \tag{4.2.11}$$

which is the exact eigenvalues obtained by solving the Schrödinger equation in a box with periodic boundary conditions.

We now proceed to obtain the classical Hamiltonian in the absence of a magnetic field which when quantised yields the discrete Laplacian, (4.2.3).

From (4.2.3) we see that the Laplacian is a combination of discrete translations in configuration space. We can therefore write the Laplacian in terms of the non-magnetic translation operators and obtain the Fourier coefficients to find the classical symbol corresponding to this operator expansion.

Writing (4.2.3) in terms of discrete translation operators gives

$$\begin{aligned} \mathrm{Op}_\hbar^W[h] &:= -\hbar^2\Delta \\ &= -\left(\frac{\hbar N}{\ell_q}\right)^2 \left(T^{(1,0),\mathbf{0}} + T^{(-1,0),\mathbf{0}} + T^{(0,1),\mathbf{0}} + T^{(0,-1),\mathbf{0}} - 4T^{\mathbf{0},\mathbf{0}}\right). \end{aligned} \tag{4.2.12}$$

From this we can immediately write down the corresponding Fourier coefficients

$$h_{\mathbf{0},\mathbf{0}} = \left(\frac{2\hbar N}{\ell_q}\right)^2, \; h_{(1,0),\mathbf{0}} = h_{(-1,0),\mathbf{0}} = h_{(0,1),\mathbf{0}} = h_{(0,-1),\mathbf{0}} = -\left(\frac{\hbar N}{\ell_q}\right), \tag{4.2.13}$$

and therefore by comparison with (3.2.26) the classical Hamiltonian which corresponds to the Laplacian is given by

$$h(\boldsymbol{q},\boldsymbol{p}) = h(\boldsymbol{p}) = \frac{\ell_p^2}{2\pi^2}\left(2 - \cos\left(\frac{2\pi p_1}{\ell_p}\right) - \cos\left(\frac{2\pi p_2}{\ell_p}\right)\right). \tag{4.2.14}$$

From this we see that $E \in [0, \frac{2\ell_p^2}{\pi^2}]$.

4.3 Landau gauge

We now look at the magnetic Weyl calculus with a vector potential in the Landau gauge which yields a constant magnetic field, with field strength b. We will now present and prove a theorem which will classify the systems which we are able to study.

Theorem 4.3.1. *The magnetic quantisation of a symbol $f \in C^\infty(\mathbb{T}^{2n})$ in the Landau gauge, $\boldsymbol{A}(\boldsymbol{q}) = (-bq^2, 0)$ can be written in terms of non-magnetic translation operators and thus, the minimal coupling can be used only when $k = M/N \in \mathbb{Z}$ for $n = 2$.*

Proof. We know from (3.3.29) that an operator in the magnetic Weyl calculus can be expressed in terms of magnetic translation operators. We therefore calculate (3.3.30) for the vector potential $\boldsymbol{A}(\boldsymbol{q}) = (-bq^2, 0)$, which yields

$$\gamma^{\mathcal{A}}\left([\boldsymbol{l}, \boldsymbol{l}-\boldsymbol{m}]\right) = -\frac{b\ell_q^2}{N^2}\int_0^1 m^1(l^2 - sm^2)\,\mathrm{d}s = -b\frac{\ell_q^2}{N^2}m^1\left(l^2 - \frac{1}{2}m^2\right), \tag{4.3.1}$$

thus,

$$\left(\mathrm{Op}_\hbar^{\mathcal{A}}[f]u\right)_{\boldsymbol{l}} = \sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^2} f_{\boldsymbol{n},\boldsymbol{m}} \mathrm{e}^{-\frac{i}{\hbar}\frac{b\ell_q^2}{N^2}m^1\left(l^2-\frac{1}{2}m^2\right)} \mathrm{e}^{\frac{2\pi i}{N}\langle \boldsymbol{n}, \boldsymbol{l}-\frac{1}{2}\boldsymbol{m}\rangle} u_{\boldsymbol{l}-\boldsymbol{m}}. \tag{4.3.2}$$

Using the flux quantisation condition $b\ell_q^2 = 2\pi\hbar M$, we find

$$\begin{aligned}\left(\mathrm{Op}_\hbar^{\mathcal{A}}[f]u\right)_{\boldsymbol{l}} &= \sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^2} f_{\boldsymbol{n},\boldsymbol{m}} \mathrm{e}^{-i\frac{2\pi}{N}\frac{M}{N}m^1\left(l^2-\frac{1}{2}m^2\right)} \mathrm{e}^{\frac{2\pi i}{N}\langle \boldsymbol{n},\boldsymbol{l}-\frac{1}{2}\boldsymbol{m}\rangle} u_{\boldsymbol{l}-\boldsymbol{m}} \\ &= \sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^2} f_{\boldsymbol{n},\boldsymbol{m}} \mathrm{e}^{\frac{2\pi i}{N}\langle \boldsymbol{n}-(0,\frac{M}{N}m^1),\boldsymbol{l}-\frac{1}{2}\boldsymbol{m}\rangle} u_{\boldsymbol{l}-\boldsymbol{m}} \\ &= \sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^2} f_{\boldsymbol{n},\boldsymbol{m}} T^{\boldsymbol{m},\boldsymbol{n}-(0,\frac{M}{N}m^1)}.\end{aligned} \tag{4.3.3}$$

Since the Fourier series is over integers, this expression is only valid if $M/N = k \in \mathbb{Z}$. Looking at the last line of (4.3.3) we see that

$$\begin{aligned}\left(\mathrm{Op}_\hbar^{\mathcal{A}}[f]u\right)_{\boldsymbol{l}} &= \sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^2} f_{\boldsymbol{n},\boldsymbol{m}} T^{\boldsymbol{m},\boldsymbol{n}-(0,km^1)} \\ &= \sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^2} f_{\boldsymbol{n}+(0,km^1),\boldsymbol{m}} T^{\boldsymbol{m},\boldsymbol{n}}.\end{aligned} \tag{4.3.4}$$

Substituting this into (3.2.26) we find

$$\begin{aligned} f(\boldsymbol{q},\boldsymbol{p}) = &\sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^2} f_{\boldsymbol{n}+(0,km^1),\boldsymbol{m}} \mathrm{e}^{2\pi i\frac{\langle \boldsymbol{n},\boldsymbol{q}\rangle}{\ell_q}-\frac{\langle \boldsymbol{m},\boldsymbol{p}\rangle}{\ell_p}} \\ &\sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^2} f_{\boldsymbol{n},\boldsymbol{m}} \mathrm{e}^{2\pi i\frac{\langle \boldsymbol{n},\boldsymbol{x}\rangle}{\ell_q}-\frac{\langle \boldsymbol{m},\boldsymbol{\xi}+(0,\frac{k\ell_p x^2}{\ell_q})\rangle}{\ell_p}}.\end{aligned} \tag{4.3.5}$$

Therefore the Fourier coefficients $f_{\boldsymbol{n}+(0,km^1),\boldsymbol{m}}$ correspond to the function $f(\boldsymbol{q},\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{q}))$ where $\boldsymbol{A}(\boldsymbol{q}) = (-Bq^2, 0)$.

We have thus found that with the Landau gauge and the flux quantisation condition, we can express the magnetic Laplace operator in terms of non-magnetic translation operators. $\square$

As we have just seen we can use the minimal coupling principle as long as our vector potential is linear and the condition $M/N \in \mathbb{Z}$ holds. This cannot be done in general with non-homogeneous magnetic fields since the minimal coupling principle is violated and we are thus unable to obtain a Hamiltonian for the non-homogeneous case. The quantum operator can still be obtained since this does not rely on the ability to use minimal coupling and thus the spectrum can be analysed.

Applying the above theorem to (4.2.14) for $k \in \mathbb{Z}$ we obtain

$$h(\boldsymbol{q}, \boldsymbol{p}) = \frac{\ell_p^2}{2\pi^2}\left(2 - \cos\left(\frac{2\pi p_1}{\ell_p} + \frac{2\pi k q^2}{\ell_q}\right) - \cos\left(\frac{2\pi p_2}{\ell_p}\right)\right) = E, \quad (4.3.6)$$

again with

$$E \in \left[0, \frac{2\ell_p^2}{\pi^2}\right]. \quad (4.3.7)$$

Before we study the kinematics of this Hamiltonian, we would first like to make a comparison to the previously studied Landau gauge with configuration space torus, [Ono08], [Ono01], which has phase space $N = \{(\boldsymbol{q}, \boldsymbol{p}); \boldsymbol{q} \in \mathbb{T}^2, \boldsymbol{p} \in \mathbb{R}^2\}$.

4.3.1 Comparison with torus configuration space

In the papers of [Ono08], [Ono01], the case of a particle in the presence of a magnetic field with a torus configuration space, $\mathbb{T}^2 := \mathbb{R}^2/(L_1\mathbb{Z} \oplus L_2\mathbb{Z})$, was studied. The results found indicated that the spectrum was that of a particle in the presence of a magnetic field with configuration space $\mathbb{R}^2$, but with an N-fold degeneracy due to translation invariance of the torus.

In the papers, the author used the usual Schrödinger operator

$$H_o = \left(i\hbar\partial_\mu - \frac{e}{c}A_\mu\right)\psi, \quad (4.3.8)$$

where A_μ are components of the gauge potential one-form $\mathcal{A} = \sum A_\mu \mathrm{d}x^\mu$. Periodic boundary conditions were introduced and the connection form was specified in each chart to ensure that the Hamiltonian, (4.3.8) is Hermitian.

The author then introduced natural units of the form

$$\hbar/m\omega = 1. \quad (4.3.9)$$

By repeated application of the boundary conditions a flux quantisation was found, i.e., $L_1 L_2 = N\pi$ which we will now show is equivalent to (3.3.26b).

In the authors convention,

$$\omega = \frac{eB}{2mc}, \tag{4.3.10}$$

and from (4.3.9) we obtain an expression for $\hbar$,

$$\hbar = \frac{eB}{2c}. \tag{4.3.11}$$

Using (3.3.26b) where $B = b$, with (4.3.11) yields

$$2\pi\left(\frac{eb}{2c}\right) M = b\ell_q^2. \tag{4.3.12}$$

Using our convention, i.e., $e = c = 1$ we obtain

$$\pi M = \ell_q^2. \tag{4.3.13}$$

We therefore see that the length parameters $\ell_q = L_1 = L_2$ and $M = N$, where N in [Ono08], [Ono01] correspond to the number of states (our N in (3.2.30)).

We have thus obtained the same flux quantisation condition, purely by allowing the magnetic translation operators to commute around the configuration space torus rather than finding a relation between the connection one forms in each chart of the torus.

We will now show that in the infinite momentum limit, $\ell_p \to \infty$, the Hamiltonian (4.3.6) reduces to that of the shifted harmonic oscillator and thus yields the system studied in [Ono08], [Ono01].

Using (4.3.6), we let $\ell_p \to \infty$ which results in small arguments for the cosine functions:

$$\begin{aligned} h(\boldsymbol{q}, \boldsymbol{p}) &= \frac{\ell_p^2}{2\pi^2}\left(2 - 1 + \frac{1}{2}\left(\frac{2\pi}{\ell_p}p_1 + \frac{2\pi B}{\ell_p}q^2\right)^2 - 1 + \frac{1}{2}\left(\frac{2\pi p_2}{\ell_p}\right)^2\right) \\ &= (p_1 + Bq^2)^2 + p_2^2. \end{aligned} \tag{4.3.14}$$

The function h was derived via the Laplacian which is the Hamiltonian of a free particle with mass 1/2.

We have therefore shown that in the infinite momentum volume limit, the discrete magnetic Laplacian yields equivalent results to that of the usual quantum mechanics on a configuration space torus.

4.3.2 Eigenvalues

4.3.2.1 Diagonalisation of the Hamiltonian matrix

One of the benefits of a torus phase space is that upon quantisation, the eigenvalues of the self-adjoint form a discrete and finite set. The operator can thus be cast in the form of a matrix and then numerically, the eigenvalues can be computed with an eigenvalue finding algorithm or as it is commonly called, via diagonalisation.

To obtain the matrix representation of the Hamiltonian we let u_{l_i} be a basis of $\mathcal{H}_{N^n}$ and thus $\{u_{\boldsymbol{l}} := u_{l_1} \otimes u_{l_2} \otimes \cdots \otimes u_{l_n}; l_1, l_2, ..., l_n = 0, 1, ..., N-1\}$ which represents N^n complex numbers of the form $u_{\boldsymbol{l}} = (u_{0,...,0,1}, ..., u_{N-1,...,N-1})^T$. Where $\boldsymbol{l}$ represents the N^n positions on the configuration space lattice. In this thesis we look at the quantisation of the Hamiltonian (4.3.6) where the corresponding operator is an $N^2 \times N^2$ matrix with matrix elements (3.1.6).

We can now write the Schrödinger equation for Hamiltonian (4.3.6) acting on a vector $u_{\boldsymbol{l}} \in \mathcal{H}_{N^2}$,

$$\left(\mathrm{Op}_\hbar^{\mathcal{A}}[h]u\right)_{\boldsymbol{l}} = E u_{\boldsymbol{l}}. \tag{4.3.15}$$

We know from before that the magnetic translations act as

$$\begin{aligned}(T_{\mathcal{A}}^{m_1,m_2,n_1,n_2} u)_{l_1,l_2} = {} & \mathrm{e}^{\frac{2\pi i}{N}(n_1 l_1 + n_2 l_2) - \frac{\pi i}{N}(n_1 m_1 + n_2 m_2) + \frac{\pi i k m_1 m_2}{N} - \frac{2\pi i k m_1 l_2}{N}} \cdot \\ & \cdot u_{l_1 - m_1, l_2 - m_2},\end{aligned} \tag{4.3.16}$$

we can therefore write down the Laplacian when $n = 2$ for a general vector potential. Due to the periodicity of the configuration space, the operator is finite dimensional and therefore we can write the eigenvector as

$$u_{\boldsymbol{l}} := u_{l_1,l_2} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix}, \quad u_i = \begin{pmatrix} u_{i,0} \\ u_{i,1} \\ u_{i,2} \\ \vdots \\ u_{i,N-1} \end{pmatrix} \quad i = 0, 1, ..., N-1 \tag{4.3.17}$$

and a general magnetic Laplacian as

$$\begin{aligned}\left(\mathrm{Op}_\hbar^{\mathcal{A}}[h]u\right)_{l_1,l_2} &= A_1(l_1,l_2)u_{l_1-1,l_2} + A_2(l_1,l_2)u_{l_1+1,l_2} + A_3(l_1,l_2)u_{l_1,l_2-1} \\ &\quad + A_4(l_1,l_2)u_{l_1,l_2+1} + A_5(l_1,l_2)u_{l_1,l_2} = Eu_{l_1,l_2}.\end{aligned} \tag{4.3.18}$$

Using (4.3.17), we obtain a Hamiltonian which is a block matrix of the form

$$\mathrm{Op}_\hbar^{\mathcal{A}}[h] = \begin{pmatrix} \mathcal{A}_0 & \mathcal{B}_0 & & \mathcal{C}_0 \\ \mathcal{C}_1 & \ddots & \ddots & \\ & \ddots & \ddots & \mathcal{B}_{N-2} \\ \mathcal{B}_{N-1} & & \mathcal{C}_{N-1} & \mathcal{A}_{N-1} \end{pmatrix}, \tag{4.3.19}$$

with

$$\mathcal{A}_j = \begin{pmatrix} A_5(j,0) & A_4(j,0) & & A_3(j,0) \\ A_3(j,1) & A_5(j,1) & \ddots & \\ & \ddots & \ddots & A_4(j,N-2) \\ A_4(j,N-1) & & A_3(j,N-1) & A_5(j,N-1) \end{pmatrix}, \tag{4.3.20}$$

$$\mathcal{B}_j = \begin{pmatrix} A_2(j,0) & & & \\ & A_2(j,1) & & \\ & & \ddots & \\ & & & A_2(j,N-1) \end{pmatrix}, \tag{4.3.21}$$

and

$$\mathcal{C}_j = \begin{pmatrix} A_1(j,0) & & & \\ & A_1(j,1) & & \\ & & \ddots & \\ & & & A_1(j,N-1) \end{pmatrix}, \tag{4.3.22}$$

where the blank entries indicate zeros. Using this result with the following operator in the Landau gauge

$$\begin{aligned}
\left(\mathrm{Op}_\hbar^{\mathcal{A}}[h]u\right)_{l_1,l_2} := &-\left(\frac{\hbar N}{\ell_q}\right)^2\left(\left(T_{\mathcal{A}}^{(1,0),\mathbf{0}}u\right)_{l_1,l_2}+\left(T_{\mathcal{A}}^{(-1,0),\mathbf{0}}u\right)_{l_1,l_2}\right.\\
&\left.+\left(T_{\mathcal{A}}^{(0,1),\mathbf{0}}u\right)_{l_1,l_2}+\left(T_{\mathcal{A}}^{(0,-1),\mathbf{0}}u\right)_{l_1,l_2}-4\left(T_{\mathcal{A}}^{\mathbf{0},\mathbf{0}}u\right)_{l_1,l_2}\right) && \text{(4.3.23a)}\\
&=-\frac{\hbar^2N^2}{\ell_q^2}\mathrm{e}^{-\frac{2\pi ikl_2}{N}}u_{l_1-1,l_2}-\frac{\hbar^2N^2}{\ell_q^2}\mathrm{e}^{\frac{2\pi ikl_2}{N}}u_{l_1+1,l_2}-\frac{\hbar^2N^2}{\ell_q^2}u_{l_1,l_2-1}\\
&-\frac{\hbar^2N^2}{\ell_q^2}u_{l_1,l_2+1}+\frac{4\hbar^2N^2}{\ell_q^2}u_{l_1,l_2} && \text{(4.3.23b)}\\
&=-\frac{\ell_p^2}{4\pi^2}\mathrm{e}^{-\frac{2\pi ikl_2}{N}}u_{l_1-1,l_2}-\frac{\ell_p^2}{4\pi^2}\mathrm{e}^{\frac{2\pi ikl_2}{N}}u_{l_1+1,l_2}-\frac{\ell_p^2}{4\pi^2}u_{l_1,l_2-1}\\
&-\frac{\ell_p^2}{4\pi^2}u_{l_1,l_2+1}+\frac{\ell_p^2}{\pi^2}u_{l_1,l_2}=Eu_{l_1,l_2} && \text{(4.3.23c)}
\end{aligned}$$

we thus obtain for the Landau gauge in two dimensions the terms

$$\begin{aligned}
&A_1(l_1,l_2)=-\frac{\ell_p^2}{4\pi^2}\mathrm{e}^{-\frac{2\pi ik}{N}l_2},\quad A_2(l_1,l_2)=-\frac{\ell_p^2}{4\pi^2}\mathrm{e}^{\frac{2\pi ik}{N}l_2},\\
&A_3(l_1,l_2)=A_4(l_1,l_2)=-\frac{\ell_p^2}{4\pi^2},\ \text{and}\ A_5(l_1,l_2)=-\frac{\ell_p^2}{\pi^2}. && \text{(4.3.24)}
\end{aligned}$$

Substituting this into the matrices defined above we obtain an expression for the Hamiltonian matrix (4.3.15). To obtain the spectrum numerically for the above operator, we fill an $N^2\times N^2$ matrix of zeros with the corresponding elements i.e., (4.3.19). With this a suitable eigenvalue finding algorithm is used to obtain the eigenvalues.

Setting $k=0$ in (4.3.23c) we find the matrix elements of the Hamiltonian operator for the free particle on the torus with the vector potential absent and with $k=0$ in (4.3.6) we find the classical Hamiltonian. Next, we compare the spectrum by diagonalising (4.3.19), with the eigenvalues obtained via EBK quantisation.

4.3.2.2 EBK-quantisation

To obtain the EBK energy spectrum we first transform the Hamiltonian to action-angle variables. Action-angle variables are obtained using Hamilton-Jacobi equation equation (2.1.28) and defining the action variables $\boldsymbol{J}$ by integrating over closed irreducible loops in phase space.

Since the position coordinates are cyclic and therefore the momentum are constant, $\boldsymbol{\alpha}$, the action variables are trivial. The action variables are defined by the closed loops,

$$J_j = \frac{1}{2\pi}\oint_{\mathcal{C}_j} p_j \mathrm{d}q^j,$$

on the LAT defined as

$$\Sigma_{LAT}^{k=0} := \{(\boldsymbol{q},\boldsymbol{p}) \in \mathbb{T}^4; p_1 = \alpha_1, p_2 = \alpha_2\}. \tag{4.3.25}$$

Due to the periodicity of the torus we obtain

$$J_1 = \frac{1}{2\pi}\oint_{\mathcal{C}_1} p_1\,\mathrm{d}q^1 = \frac{1}{2\pi}\int_0^{\ell_q} \alpha_1\,\mathrm{d}q^2 = \frac{\alpha_1 \ell_q}{2\pi}, \tag{4.3.26}$$

$$J_2 = \frac{1}{2\pi}\oint_{\mathcal{C}_2} p_2\,\mathrm{d}q^2 = \frac{1}{2\pi}\int_0^{\ell_q} \alpha_2\,\mathrm{d}q^2 = \frac{\alpha_2 \ell_q}{2\pi}. \tag{4.3.27}$$

The Hamiltonian can thus be written as

$$H(J_1, J_2) = \frac{\ell_p^2}{2\pi^2}\left(2 - \cos\left(\frac{4\pi^2 J_1}{\ell_p \ell_q}\right) - \cos\left(\frac{4\pi^2 J_2}{\ell_p \ell_q}\right)\right) = E. \tag{4.3.28}$$

Therefore using EBK-quantisation, with $\boldsymbol{\alpha} = (0,0)$ and the torus phase space quantisation condition, $2\pi\hbar N = \ell_p \ell_q$, we find

$$E_{m_1,m_2} = \frac{\ell_p^2}{2\pi^2}\left(2 - \cos\left(\frac{2\pi m_1}{N}\right) - \cos\left(\frac{2\pi m_2}{N}\right)\right). \tag{4.3.29}$$

Below we show a plot of the eigenvalues computed from (4.3.29) and via the diagonalisation of (4.3.23c) when $k = 0$.

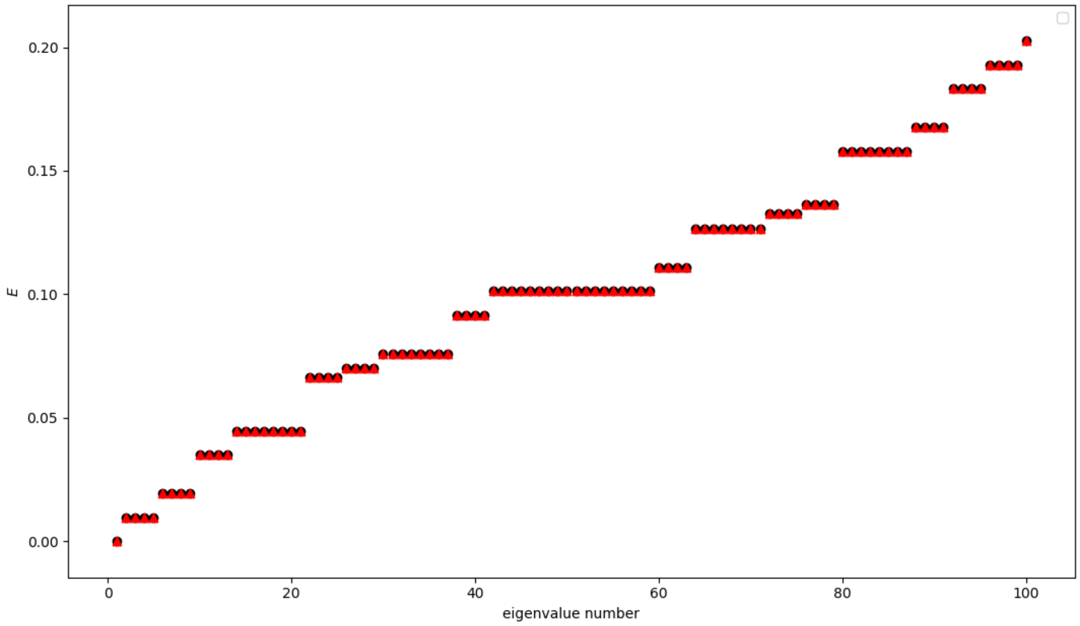

Figure 4.1: A comparison of the spectrum obtained via EBK, i.e., (4.3.29) and via the diagonalisation of (4.3.23c). The red markers correspond to the eigenvalues of (4.3.23c) and the black markers to (4.3.29).

The eigenvalues computed using EBK and from diagonalising the Hamiltonian matrix agree with one another, exactly for some eigenvalues, but for most the error in the difference is 10^{-16}.

Using this method to fill the matrix Hamiltonian we plot the spectrum for (4.3.23c) with various values of integer values of k with the corresponding classical Hamiltonian written as (4.3.6).

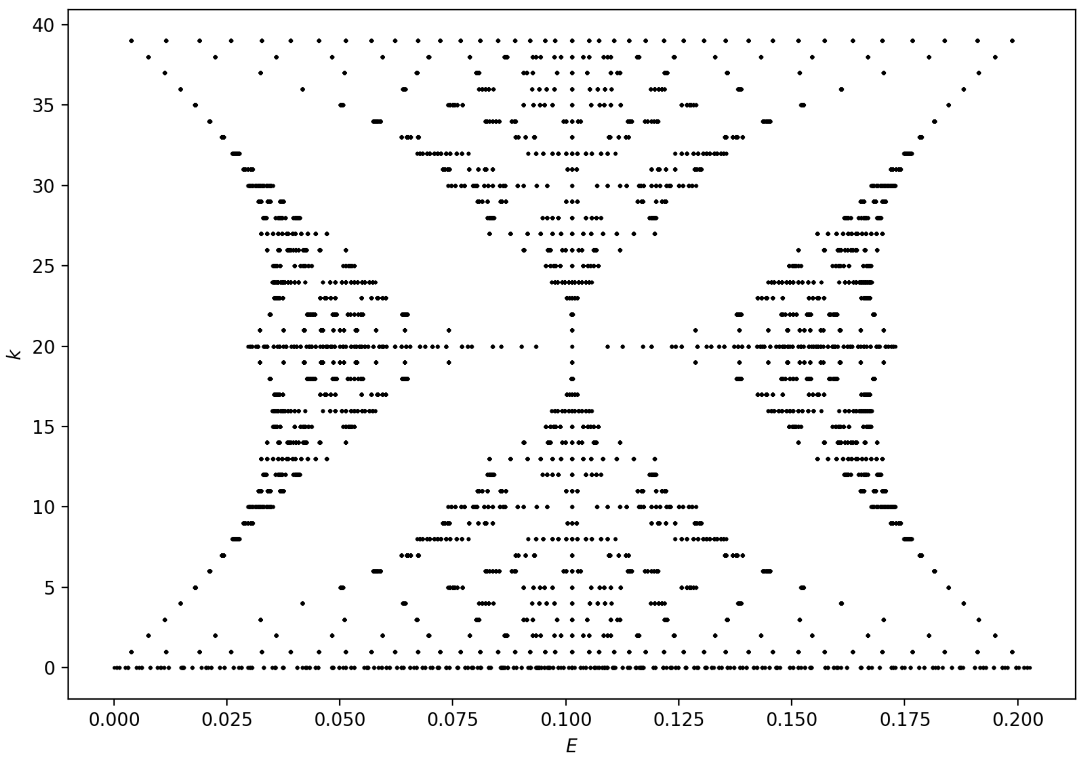

Figure 4.2: The spectrum of (4.3.23c) for $N = 40$ and $\ell_p = \ell_q = 1$ for various values of integer values of k.

From the plot above we see that $E \in [0, \frac{2}{\pi^2}]$ which agrees with (4.3.7). The spectrum above is similar to the regularised spectrum in [Hof76] (see figure 3). We see from, (4.3.23c), the spectrum is periodic with respect to k, i.e., $E(k + N) = E(k)$. Comparing the case of $k = 0$ and $k = 1$ we see that when a magnetic field is introduced a degeneracy of eigenvalues is present. Since the spectrum is periodic with respect to N the degeneracy is also periodic with period N.

The degeneracy condition can be thought of as a discrete form of a condition similar to that of an energy surface. The degeneracy at energy E, denoted by $d_E \in \mathbb{N}$, can be written as

$$d_E := \#\{(m_1, m_2) \in \mathbb{Z}_N^2 | E_{m_1,m_2} = E\}. \tag{4.3.30}$$

We now plot the sorted energy eigenvalues (eigenvalue number of 1 corresponds to the ground state) for different k values.

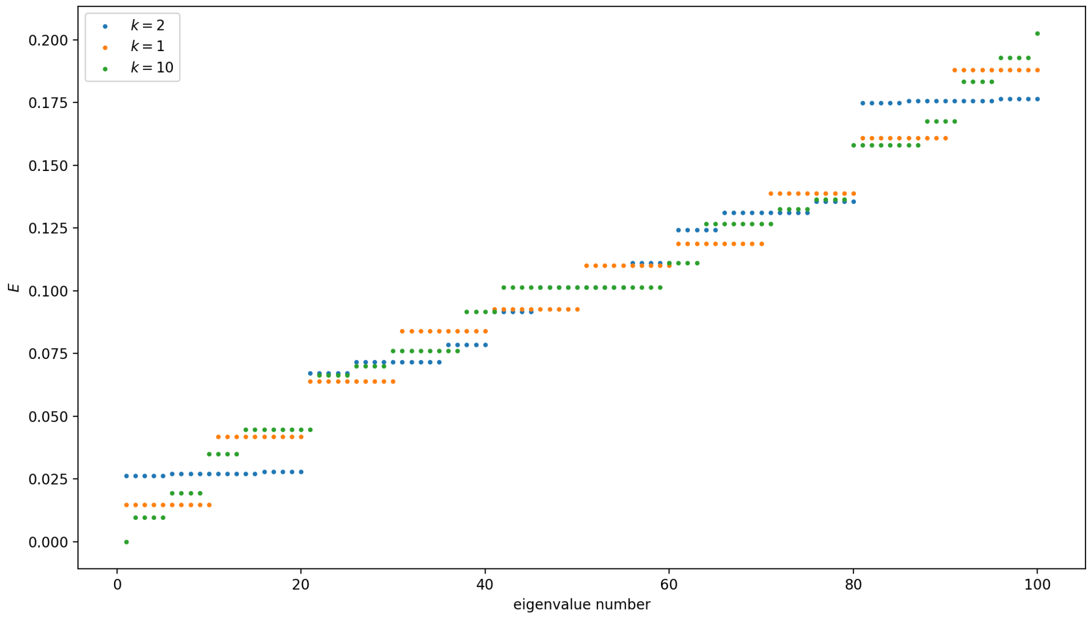

Figure 4.3: The spectrum of (4.3.23c) for $N = 10$ and $\ell_p = \ell_q = 1$ for $k = 1, 2, 10$.

We see that when $k = 1$ we have an N-fold degeneracy. But for $k = 2$, this N-fold degeneracy is broken. We have five levels with a 10-fold degeneracy and the remaining ten levels with a 5-fold degeneracy. For $k = 10$, we have two levels with no degeneracy, one level with an 18-fold degeneracy, four levels with a 8-fold degeneracy, and 12 levels with a 4-fold degeneracy.

We now plot the different eigenvalue degeneracies with respect to k. For graphs with one point, this implies that all eigenvalues have the same degeneracy. For example, from figure 4.5, the case of $k = 1$ has one point at $(1, 10)$, this implies that each eigenvalue has a 10-fold degeneracy. Figure 4.5 shows the degeneracy for fewer values of the magnetic field. We let $n = (n_1, ..., n_s)$ be the number of eigenvalues with degeneracy $d = (d_1, ..., d_s)$, where s is the total number of different degeneracies. We see that

$$\langle n, d\rangle = \sum_{i=1}^{s} n_i d_i = N^2, \tag{4.3.31}$$

For example, when $k = 10$ and $N = 10$, we have $i = 4$ or four different degeneracies. For the first degeneracy we have two eigenvalues with degeneracy one, i.e., $n_1 = 2$ and $d_1 = 1$. For the second degeneracy ($i = 2$) we have one eigenvalue with an 18-fold degeneracy or, $n_2 = 1$ and $d_2 = 18$ for the third degeneracy, we have four eigenvalues with an 8-fold degeneracy i.e., $n_3 = 4$ and $d_3 = 8$ and finally for the last degeneracy we have 12 eigenvalues with a 4-fold degeneracy or $n_4 = 12$ and $d_4 = 4$. Computing (4.3.31) we find

$$\langle n, d\rangle = 2 \cdot 1 + 1 \cdot 18 + 4 \cdot 8 + 12 \cdot 4 = 100.$$

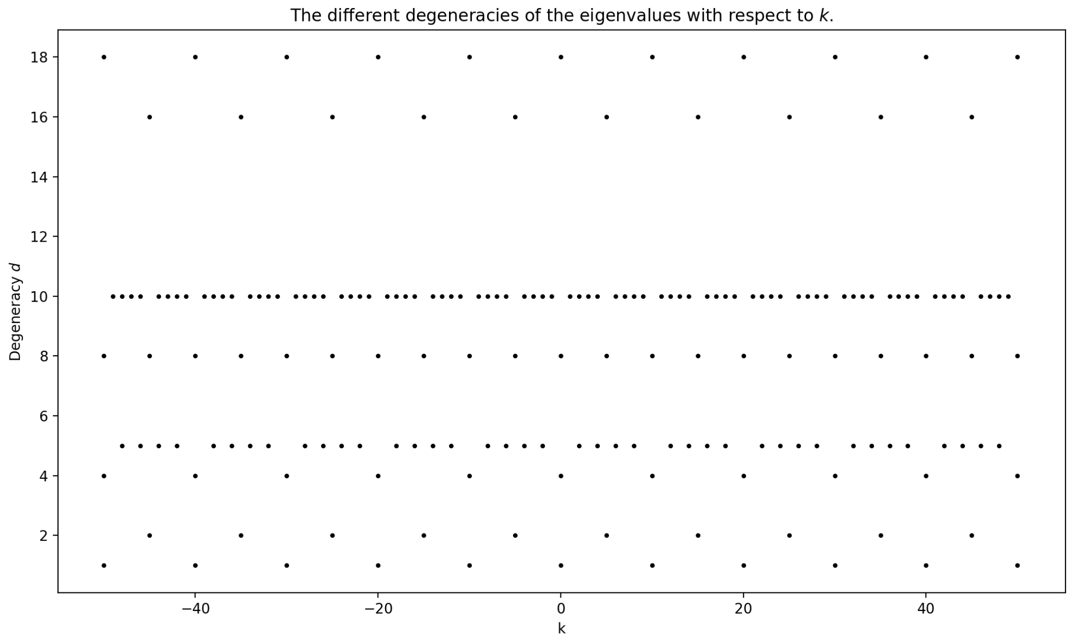

Figure 4.4: A plot of the degeneracies, d of the spectrum for (4.3.23c) for $N = 10$ and $\ell_p = \ell_q = 1$ for various integer values of k. From the graph the degeneracies are periodic with respect to k.

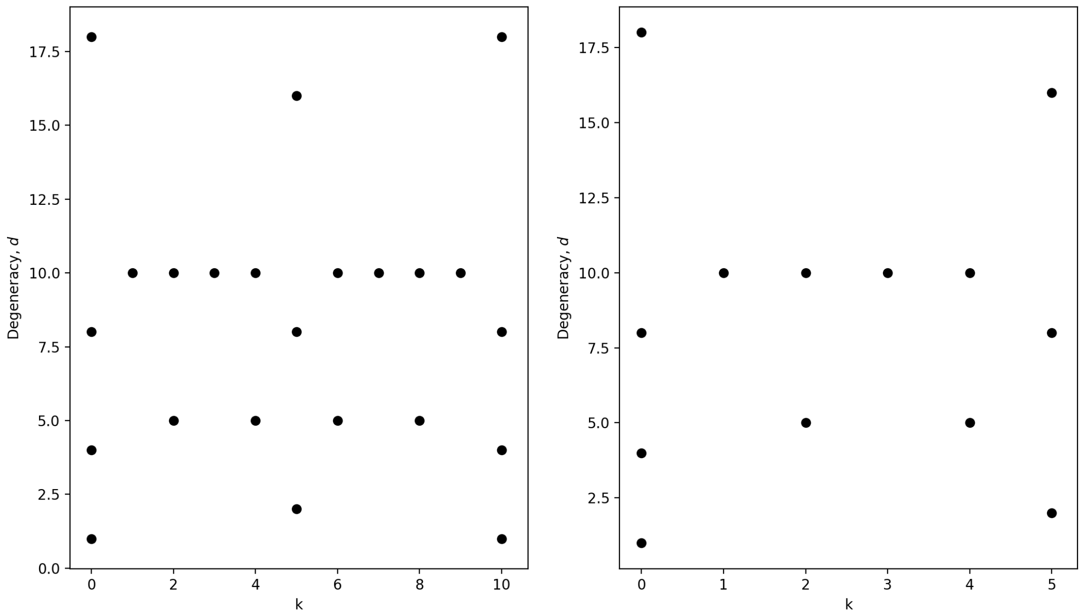

Figure 4.5: A figure showing the degeneracies, d of the spectrum for (4.3.23c) for $N = 10$ and $\ell_p = \ell_q = 1$ for various values of k.

Below we plot the degeneracy for different values of N and see that the eigenvalue degeneracy is periodic in k.

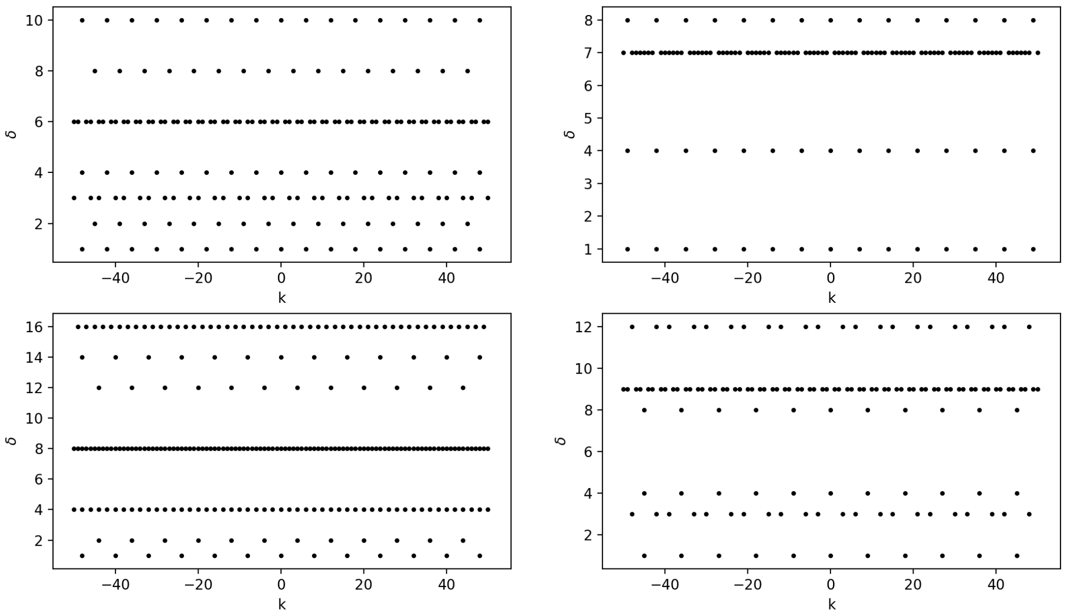

Figure 4.6: We have a plot of the degeneracies of the spectrum for (4.3.23c) for $N = 6$ (top left), $N = 7$ (top right), $N = 8$ (bottom left), and $N = 9$ (bottom right) with $\ell_p = \ell_q = 1$.

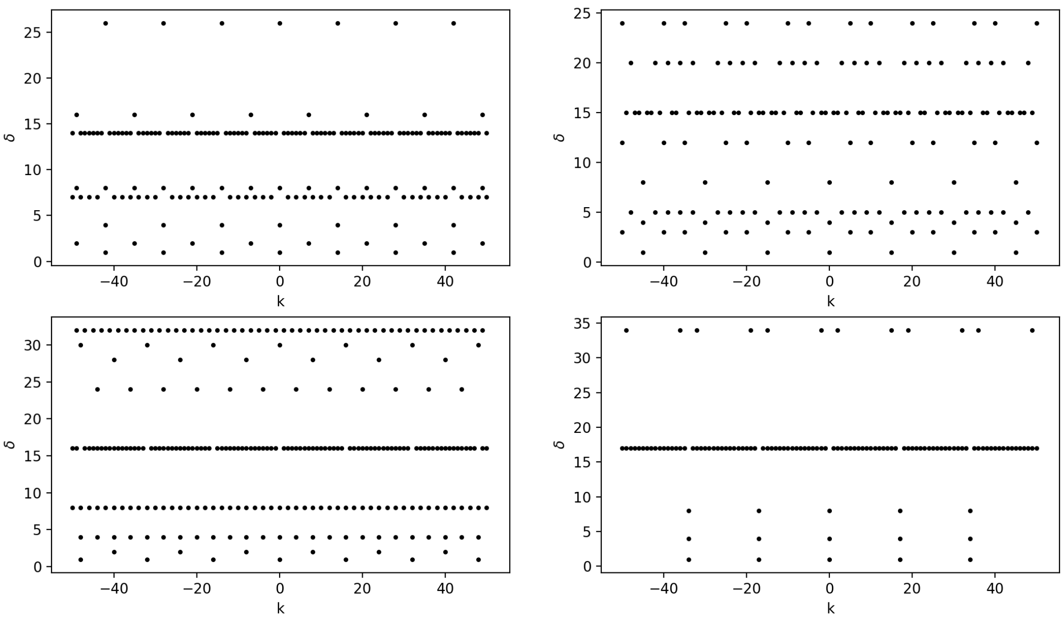

Figure 4.7: The degeneracies of the spectrum for (4.3.23c) for for $N = 14$ (top left), $N = 15$ (top right), $N = 16$ (bottom left), and $N = 17$ (bottom right) with $\ell_p = \ell_q = 1$, again for various values of k.

Below is a plot of when $B = k \in \mathbb{R}$, such that it takes all values in an interval rather than integer values. We see with non-integer values that the Hofstadter spectrum appears, thus, the inclusion of an irrational magnetic field strength produces the self-similar structure of fractals into the spectrum, see [Hof76].

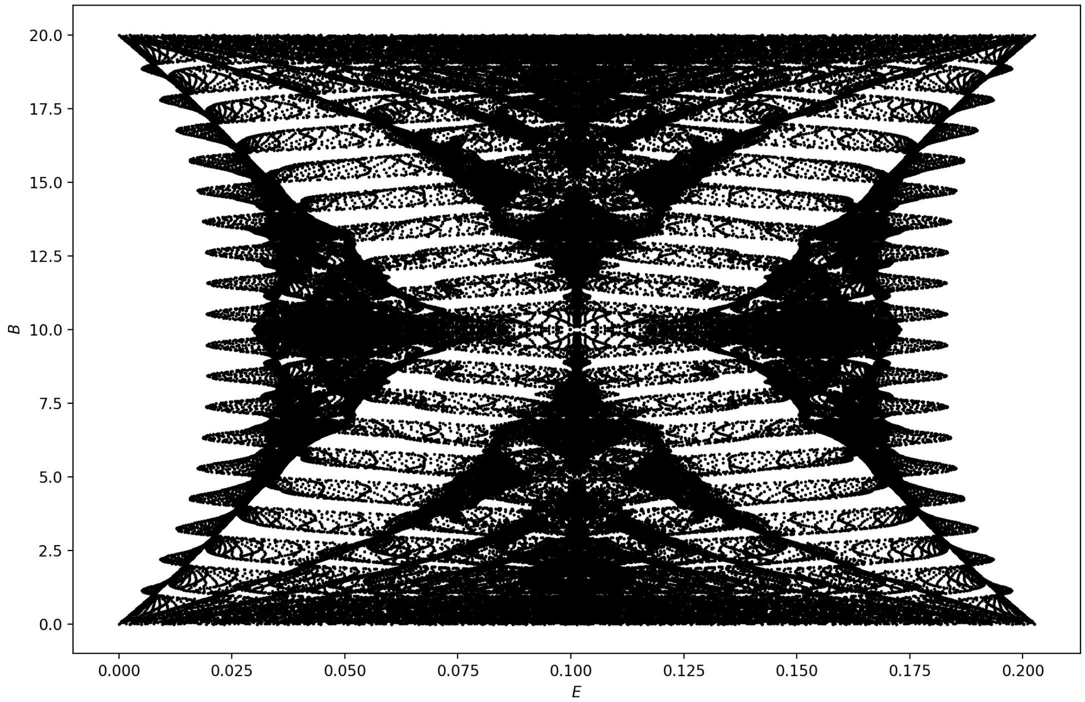

Figure 4.8: We have a plot for the spectrum of (4.3.23c) for $N = 20$ for $B = k \in [0, 30]$.

To make a comparison with [Hof76] by instead plotting the energy eigenvalues with the parameter α, we replace the parameters in [Hof76] with the parameters in this thesis which yields

$$\alpha = \frac{a^2 He}{2\pi\hbar c} = \frac{b\ell_q^2}{2\pi\hbar N^2}, \tag{4.3.32}$$

where the left hand side is the notation used in [Hof76]. Using (4.3.23c) and (4.3.32), we find a an equation similar to that of Harper, [Har55]. This produces the plot 4.8 with the x-axis ranging from 0 to 1. By continuously changing the magnetic field (figure 4.8) ones sees the characteristic Hofstadter's butterfly spectrum, exhibiting a self similar structure.

4.4 Non-homogeneous magnetic field

We now use (4.3.18) to numerically obtain the spectrum of a Laplacian with a non-homogenous magnetic field. We pick for the vector potential,

$$A_{\text{non}}(q^1, q^2) = \left(A_1(q^1, q^2), A_2(q^1, q^2)\right). \tag{4.4.1}$$

where

$$\begin{aligned} A_1(q^1, q^2) = &\frac{B\epsilon\ell_q^4 \cos\left(\frac{2\pi(q^1+q^2)}{\ell_q}\right)}{2\pi^2} + \frac{B\epsilon\ell_q^4 \cos\left(\frac{4\pi(q^1+q^2)}{\ell_q}\right)}{8\pi^2} \\ &+ \frac{B\epsilon\ell_q^4 \cos\left(\frac{2\pi(2q^1+q^2)}{\ell_q}\right)}{4\pi^2} + \frac{B\epsilon\ell_q^4 \cos\left(\frac{2\pi(q^1+2q^2)}{\ell_q}\right)}{4\pi^2} - bq^2 \end{aligned} \tag{4.4.2}$$

and

$$\begin{aligned} A_2(q^1, q^2) = &\frac{B\epsilon\ell_q^4 \sin\left(\frac{2\pi(q^1+q^2)}{\ell_q}\right)}{2\pi^3} + \frac{B\epsilon\ell_q^4 \sin\left(\frac{4\pi(q^1+q^2)}{\ell_q}\right)}{16\pi^3} \\ &+ \frac{B\epsilon\ell_q^4 \sin\left(\frac{2\pi(2q^1+q^2)}{\ell_q}\right)}{8\pi^3} + \frac{B\epsilon\ell_q^4 \sin\left(\frac{2\pi(q^1+2q^2)}{\ell_q}\right)}{4\pi^3}. \end{aligned} \tag{4.4.3}$$

The specific form of the potential was chosen so that a linear canonical transformation could not separate the vector potential into a product of two terms. The form is loosely based on the Fourier series for the Henon-Heiles potential,

$$V(q^1, q^2) = \frac{1}{2}((q^1)^2 + (q^2)^2) + (q^1)^2(q^2) - \frac{1}{3}(q^2)^3. \tag{4.4.4}$$

Any vector potential can be chosen so long as it is periodic with respect to the configuration space torus, see [Fio12] for a treatment of quantum mechanics on $\mathbb{T}^{2n}$ and $\mathbb{R}^{2n}$ and the available vector potentials which are allowed. When $\epsilon \to 0$ we arrive at the homogeneous case, therefore, we can study what affect turning on a non-uniformity has on the spectrum. Increasing the non-uniformity will increase the complexity of the dynamics in the presence of charged particles, possibly breaking symmetries which will affect the spectrum.

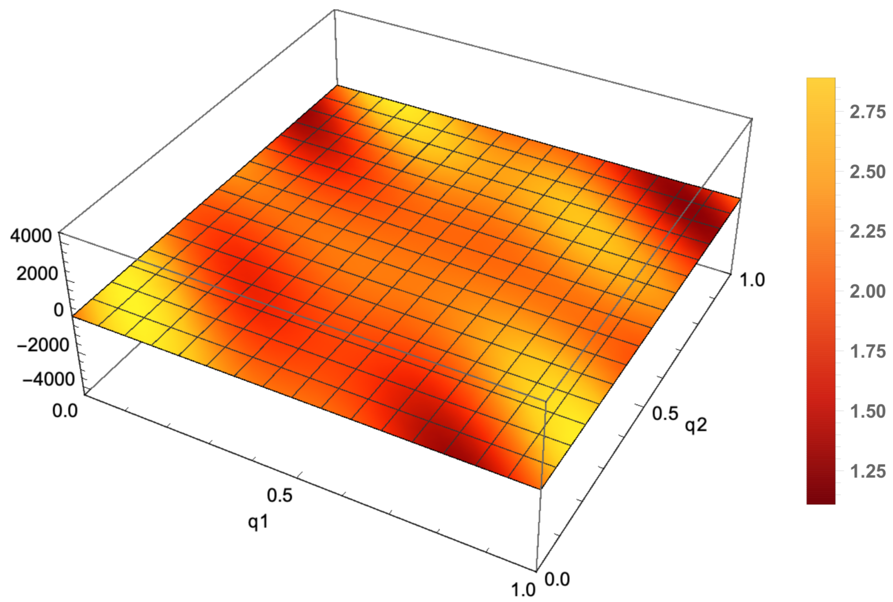

Figure 4.9: The magnetic field strength, $B = \nabla \times \boldsymbol{A}$, is plotted for $\epsilon = 0.5$, $b = 2$, and $\ell_q = 1$ using (4.4.1).

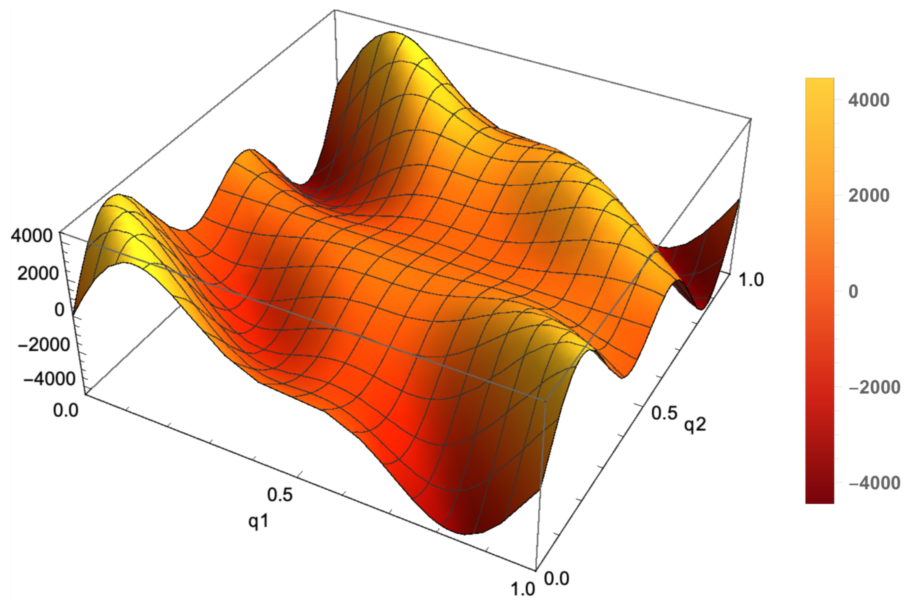

Figure 4.10: The magnetic field strength is plotted but for $\epsilon = 2500$, $B = 2$, and $\ell_q = 1$ using (4.4.1). Increasing the non-uniformity has increased the size of the peaks and troughs, which result in complex orbits for charged particles.

Using the magnetic Laplacian, (4.3.23), with (4.4.1) along with $\Gamma^{\mathcal{A}}([q, q]) =$

0 for all $\mathcal{A}$, yields

$$
\begin{aligned}
\left(\mathrm{Op}_\hbar^{\mathcal{A}_{\mathrm{non}}}[-\Delta]u\right)_{l_1,l_2} = &-\frac{\ell_p^2}{\pi^2}u_{l_1,l_2} - \frac{\ell_p^2}{4\pi^2}e^{-i\epsilon B_1(l_1,l_2)-\frac{2\pi i B\ell_q}{\ell_p N}l_2}u_{l_1-1,l_2} \\
&-\frac{\ell_p^2}{4\pi^2}e^{-i\epsilon B_2(l_1,l_2)}u_{l_1,l_2-1} - \frac{\ell_p^2}{4\pi^2}e^{-i\epsilon B_3(l_1,l_2)}u_{l_1,l_2+1} \\
&-\frac{\ell_p^2}{4\pi^2}e^{-i\epsilon B_4(l_1,l_2)+\frac{2\pi i B\ell_q}{\ell_p N}l_2}u_{l_1+1,l_2},
\end{aligned}
\tag{4.4.5}
$$

where,

$$
\begin{aligned}
B_1(l_1,l_2) = &\frac{B\ell_q^4 N\sin\left(\frac{2\pi(l_1+l_2)}{N}\right)}{2\pi^2\ell_p} + \frac{B\ell_q^4 N\sin\left(\frac{4\pi(l_1+l_2)}{N}\right)}{16\pi^2\ell_p} \\
&+\frac{B\ell_q^4 N\sin\left(\frac{2\pi(2l_1+l_2)}{N}\right)}{8\pi^2\ell_p} + \frac{B\ell_q^4 N\sin\left(\frac{2\pi(l_1+2l_2)}{N}\right)}{4\pi^2\ell_p} \\
&-\frac{B\ell_q^4 N\sin\left(\frac{2\pi(l_1+l_2+1)}{N}\right)}{2\pi^2\ell_p} - \frac{B\ell_q^4 N\sin\left(\frac{2\pi(l_1+2l_2+1)}{N}\right)}{4\pi^2\ell_p} \\
&-\frac{B\ell_q^4 N\sin\left(\frac{2\pi(2l_1+l_2+2)}{N}\right)}{8\pi^2\ell_p} - \frac{B\ell_q^4 N\sin\left(\frac{4\pi(l_1+l_2+1)}{N}\right)}{16\pi^2\ell_p},
\end{aligned}
\tag{4.4.6}
$$

$$
\begin{aligned}
B_2(l_1,l_2) = &\frac{B\ell_q^4 N\cos\left(\frac{2\pi(l_1+l_2+1)}{N}\right)}{2\pi^3\ell_p} + \frac{B\ell_q^4 N\cos\left(\frac{4\pi(l_1+l_2+1)}{N}\right)}{32\pi^3\ell_p} \\
&+\frac{B\ell_q^4 N\cos\left(\frac{2\pi(2l_1+l_2+1)}{N}\right)}{8\pi^3\ell_p} + \frac{B\ell_q^4 N\cos\left(\frac{2\pi(l_1+2l_2+2)}{N}\right)}{8\pi^3\ell_p} \\
&-\frac{B\ell_q^4 N\cos\left(\frac{2\pi(l_1+l_2)}{N}\right)}{2\pi^3\ell_p} - \frac{B\ell_q^4 N\cos\left(\frac{2\pi(2l_1+l_2)}{N}\right)}{8\pi^3\ell_p} \\
&-\frac{B\ell_q^4 N\cos\left(\frac{2\pi(l_1+2l_2)}{N}\right)}{8\pi^3\ell_p} - \frac{B\ell_q^4 N\cos\left(\frac{4\pi(l_1+l_2)}{N}\right)}{32\pi^3\ell_p},
\end{aligned}
\tag{4.4.7}
$$

$$
\begin{aligned}
B_3(l_1,l_2) = &\frac{B\ell_q^4 N \cos\left(\frac{2\pi(l_1+l_2-1)}{N}\right)}{2\pi^3\ell_p} + \frac{B\ell_q^4 N \cos\left(\frac{4\pi(l_1+l_2-1)}{N}\right)}{32\pi^3\ell_p} \\
&+ \frac{B\ell_q^4 N \cos\left(\frac{2\pi(2l_1+l_2-1)}{N}\right)}{8\pi^3\ell_p} + \frac{B\ell_q^4 N \cos\left(\frac{2\pi(l_1+2l_2-2)}{N}\right)}{8\pi^3\ell_p} \\
&- \frac{B\ell_q^4 N \cos\left(\frac{2\pi(l_1+l_2)}{N}\right)}{2\pi^3\ell_p} - \frac{B\ell_q^4 N \cos\left(\frac{2\pi(2l_1+l_2)}{N}\right)}{8\pi^3\ell_p} \\
&- \frac{B\ell_q^4 N \cos\left(\frac{2\pi(l_1+2l_2)}{N}\right)}{8\pi^3\ell_p} - \frac{B\ell_q^4 N \cos\left(\frac{4\pi(l_1+l_2)}{N}\right)}{32\pi^3\ell_p},
\end{aligned}
\tag{4.4.8}
$$

$$
\begin{aligned}
B_4(l_1,l_2) = &\frac{B\ell_q^4 N \sin\left(\frac{2\pi(l_1+l_2)}{N}\right)}{2\pi^2\ell_p} + \frac{B\ell_q^4 N \sin\left(\frac{4\pi(l_1+l_2)}{N}\right)}{16\pi^2\ell_p} \\
&+ \frac{B\ell_q^4 N \sin\left(\frac{2\pi(2l_1+l_2)}{N}\right)}{8\pi^2\ell_p} + \frac{B\ell_q^4 N \sin\left(\frac{2\pi(l_1+2l_2)}{N}\right)}{4\pi^2\ell_p} \\
&- \frac{B\ell_q^4 N \sin\left(\frac{2\pi(l_1+l_2-1)}{N}\right)}{2\pi^2\ell_p} - \frac{B\ell_q^4 N \sin\left(\frac{2\pi(l_1+2l_2-1)}{N}\right)}{4\pi^2\ell_p} \\
&- \frac{B\ell_q^4 N \sin\left(\frac{2\pi(2l_1+l_2-2)}{N}\right)}{8\pi^2\ell_p} - \frac{B\ell_q^4 N \sin\left(\frac{4\pi(l_1+l_2-1)}{N}\right)}{16\pi^2\ell_p}.
\end{aligned}
\tag{4.4.9}
$$

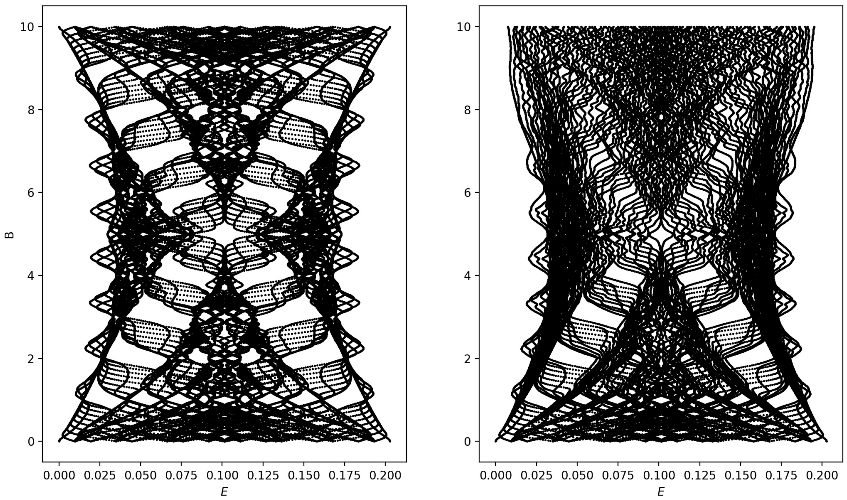

Figure 4.11: The eigenvalues of (4.4.5) with $\epsilon = 0.0$ (top), $\epsilon = 4$ (bottom), and $N = 10$ with the magnetic strength, B, plotted on the y-axis.

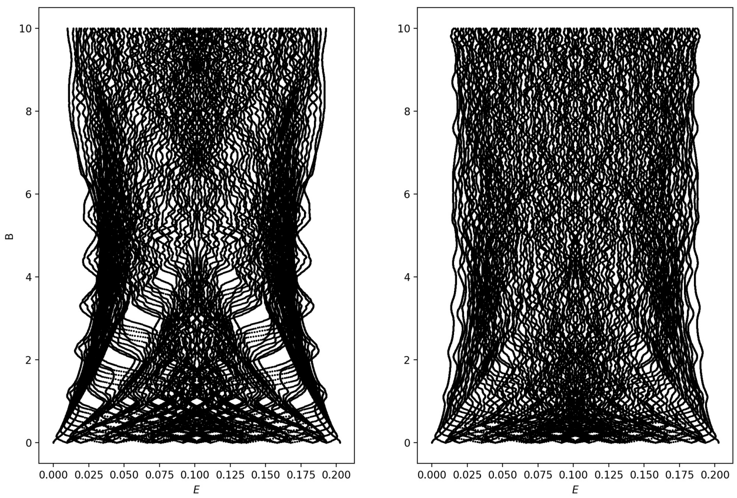

Figure 4.12: The eigenvalues of (4.4.5) with $\epsilon = 6.0$ (top), $\epsilon = 15.0$ (bottom), and $N = 15$ with the magnetic strength, B on the y-axis.

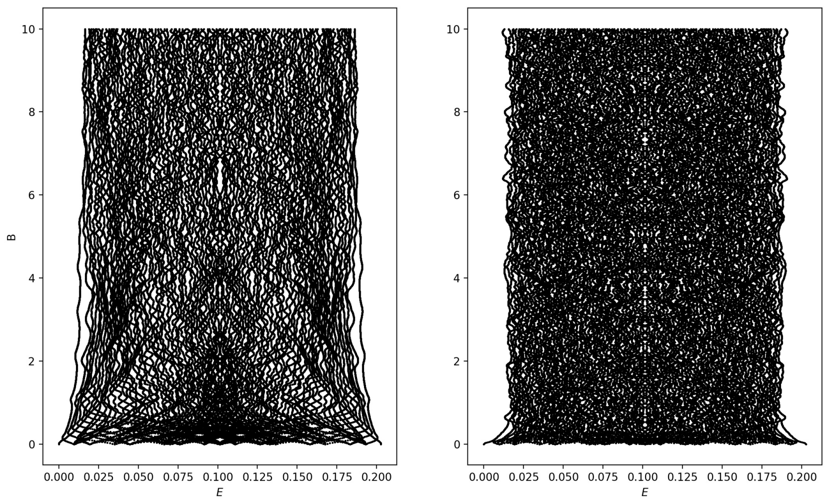

Figure 4.13: The eigenvalues of (4.4.5) with $\epsilon = 10.0$ (top), $\epsilon = 25.0$ (bottom), and $N = 15$ with the magnetic strength, B on the y-axis.

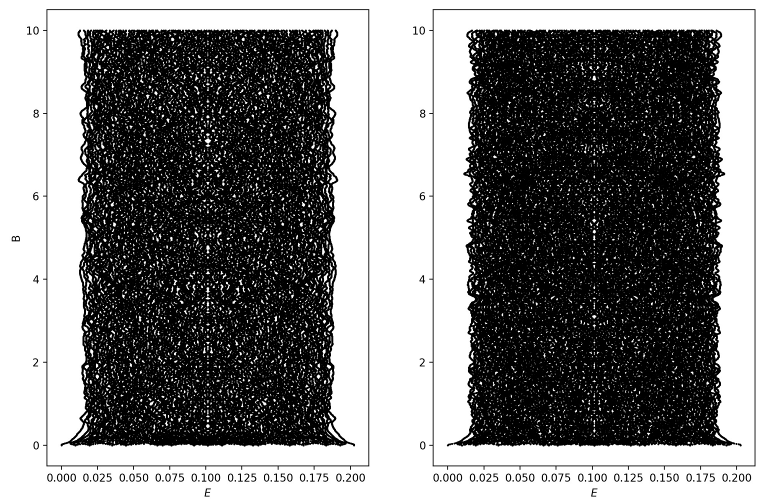

Figure 4.14: The eigenvalues of (4.4.5) with $\epsilon = 100.0$, 2500.0, and $N = 15$ with the magnetic strength, B on the y-axis.

We see that the "butterfly" spectrum slowly gets destroyed as ϵ increases until the spectrum with respect to B looks like noise. Notice also the energy $E \in [0, 2\ell_p^2/\pi^2]$. Previously we found degeneracies in the eigenvalues shown by figure 4.3; we now look at what effect increasing the non-homogeneity has on the spectrum and the degeneracy of the eigenvalues.

Below we have a plot of the eigenvalues with respect to the eigenvalue number (again $N = 10$) for different values of ϵ.

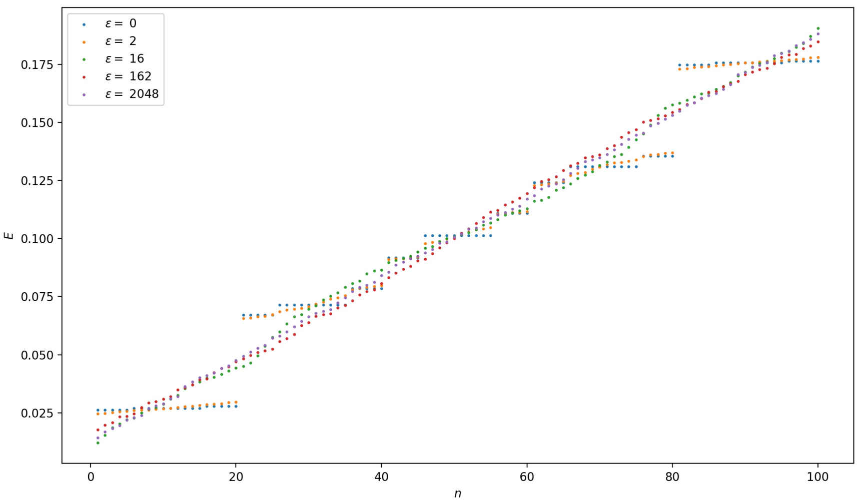

Figure 4.15: The energy spectrum of (4.4.5) for $B = 2$ and $N = 10$ as ϵ is increased. As ϵ is increased we see that the degeneracy is lost and becomes a linear function of n.

It is shown that by increasing ϵ the eigenvalues lose their degeneracy. When degeneracies are present, certain spectral statistics yield inconclusive results for example, the eigenvalue counting function obtains large contributions from the degenerate eigenvalues. The break down of the system's degeneracies thus aids the eigenvalue analysis. The breaking of the degeneracy also implies a loss of symmetry since, a degenerate spectrum hints to a symmetry in the system.

In figures 4.16, 4.17, 4.18 we plot the histogram of eigenvalues to obtain the distribution of the energy spectrum.

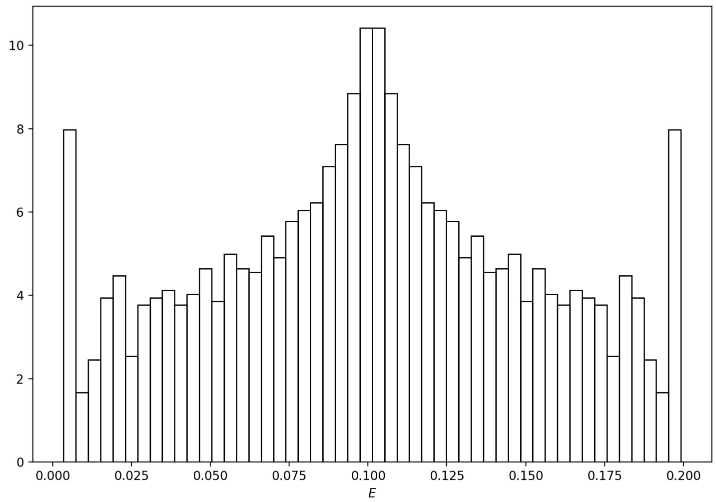

Figure 4.16: This figure shows a histogram for the the eigenvalues of (4.4.5) when $\epsilon = 20$, $B = 2$ and $N = 54$.

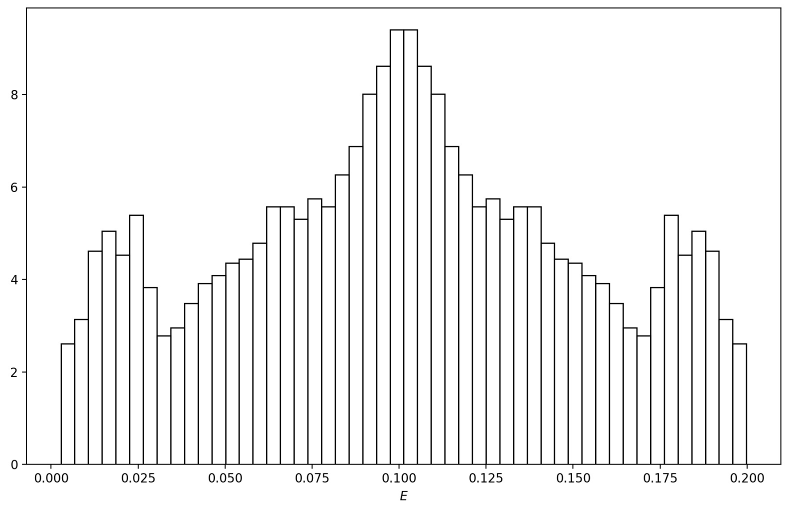

Figure 4.17: This figure shows a histogram for the eigenvalues of (4.4.5) when $\epsilon = 200$, $B = 2$ and $N = 54$.

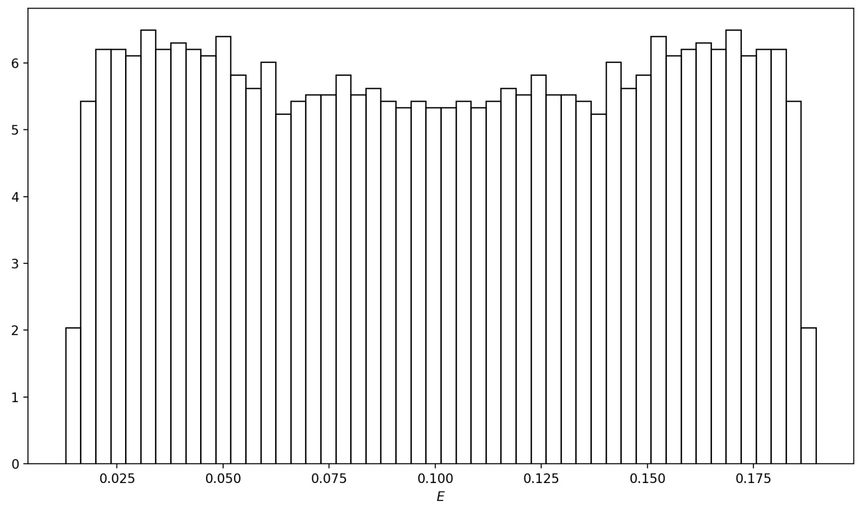

Figure 4.18: This figure shows a histogram for the eigenvalues of (4.4.5) when $\epsilon = 200,000$, $B = 2$ and $N = 54$.

In the next section we will be interested in studying the eigenvalue statistics of each system mentioned above using tests known in random matrix theory. These tests have been conjectured to shed light on the type of system in question whether integrable or chaotic.

4.5 Spectral statistics

In the field of quantum chaos a basic assumption is that one should be able to reveal signatures of classical chaos within the eigenvalue distribution of the quantum Hamiltonian, $\mathrm{Op}_\hbar^{\mathcal{A}}[h]$. The distribution of the eigenvalues should follow certain universal rules depending on whether the underlying classical system is chaotic or not. One test is to search for properties in the level spacing distribution.

We first 'unfold' the spectrum, this is a normalisation of the spacings which is done so that every spectrum studied has the same mean separation and thus can be studied on equal grounds (universality).

From diagonalisation of our Hamiltonian matrix we obtain a list of eigenvalues which we sort, and label as Spec $\left(\mathrm{Op}_{\hbar}^{\mathcal{A}}[h]\right) = \{E_0, E_1, \cdots, E_{N^n-1}\}$, where $E_0 \leq E_1 \leq \cdots \leq E_{N^n-1}$. The spacing between the eigenvalues is defined as $s_i = E_{i+1} - E_i$, which are elements of the set $\mathcal{E}_s := \{s_0, \cdots, s_{N^n-2}\}$, with the mean spacing, $D = \frac{1}{N^n-1}\sum_{i=0}^{N^n-2} s_i$. We use the relative spacing $t_i = s_i/D \in \mathcal{E}_t := \{t_0, t_1, \cdots, t_{N^n-2}\}$ to obtain the probability density function $p(t)$ which is found by placing the relative spacings into m bins. The distribution is then compared to that of an appropriate random matrix ensemble, where the choice depends on the symmetry of the system.

It has been conjectured [MVB77b], that the relative spacing of the eigenvalues of the quantisation of an integrable system obeys the Poisson distribution,

$$p_P(t) = \mathrm{e}^{-t}. \tag{4.5.1}$$

For systems without invariance under time reversal symmetry, the relative spacing distribution follows that of Gaussian unitary ensemble (GUE) or approximately the Circular orthogonal ensemble, such systems include atoms in an external magnetic field. A system with relative spacing distribution that of the Gaussian orthogonal ensemble (GOE), [OBS84], or the GUE, [LHC14, USH95], is said to be chaotic. The GUE relative spacing distribution has the following form

$$p_{GUE}(t) = \frac{32}{\pi^2} t^2 \mathrm{e}^{-\frac{4}{\pi}t^2}. \tag{4.5.2}$$

Below we plot the relative spacings of three systems,

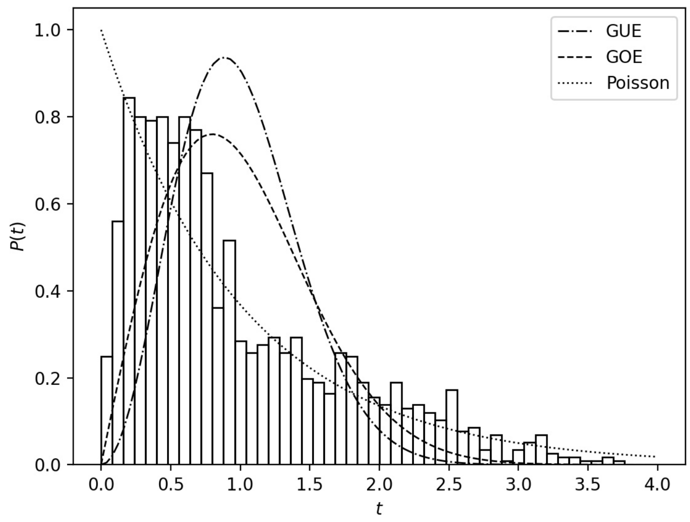

Figure 4.19: The relative level spacing of (4.4.5) for $B = 2$ and $\epsilon = 20$. From this the level spacing is close to that of the Poissonian distribution, hinting that with low non-homogeneity, the distribution indicates that of a integrable system.

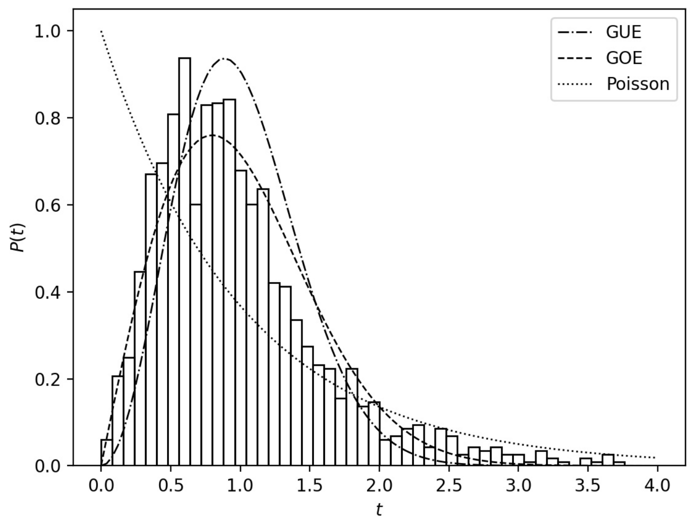

Figure 4.20: The relative level spacing of (4.4.5) for $B = 2$ and $\epsilon = 200$. From this the level spacing is moving away from the Poissonian distribution to the GOE, hinting that with as the non-homogeneity is turned higher, the distribution indicates that the system is changing the symmetry of the Hamiltonian and is becoming chaotic.

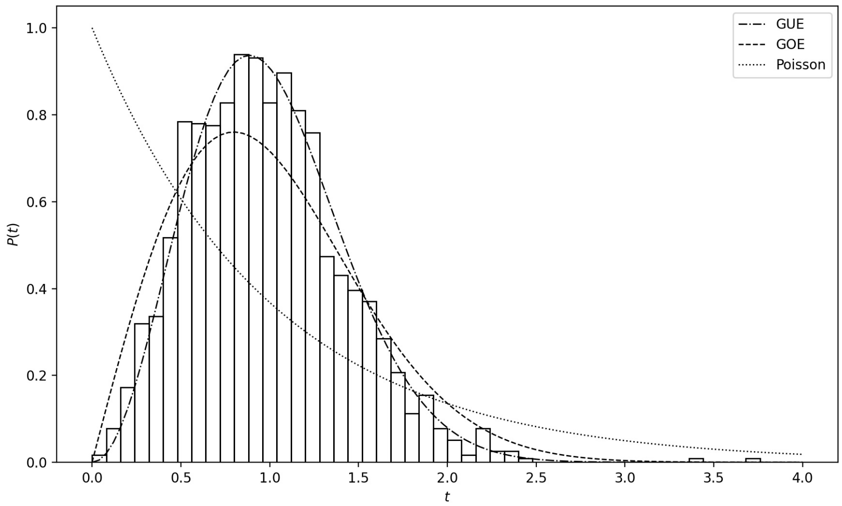

Figure 4.21: The relative level spacing of (4.4.5) for $B = 2$ and $\epsilon = 200,000$. From this the level spacing has moved away from the Poissonian distribution, hinting that with a high non-homogeneity, the distribution indicates that the system is chaotic and is not time reversal symmetric.

Another statistic is the cumulative distribution which has the following form

$$I(t) := \int_0^t p(x)\,\mathrm{d}x = \lim_{N\to\infty} \frac{\#\{i; t_i \in \mathcal{E}_t \leq t\}}{N^n}, \tag{4.5.3}$$

where $\mathcal{E}_t$ is the set of all relative spacings for the spectrum $\mathrm{Op}_\hbar^{\mathcal{A}}[h]$. This has benefits over the corresponding spacing distribution since it avoids binning and yields a smoother curve.

As mentioned earlier, degeneracy causes problems for the study of spectral statistics since the level spacings acquire a large contribution from a spacing of 0. For example, consider the following spectrum of a hypothetical Hamiltonian, $\hat{h}$, of 9 energy eigenvalues which are 3-fold degenerate, i.e., $\mathrm{Spec}(\hat{h}) = \{E_1, E_1, E_1, E_2, E_2, E_2, E_3, E_3, E_3\}$ where $E_1 < E_2 < E_3$. We now unfold the spectrum to obtain the spacings which we label by the set $\mathcal{E}_s = \{0, 0, E_2 - E_1, 0, 0, E_3 - E_2, 0, 0\}$. The mean spacing is calculated from this as is found to be $D = \frac{(E_3 - E_1)}{8}$. The set of normalised spacings defined as $\mathcal{E}_t$, is therefore

$$\mathcal{E}_t = \left\{0, 0, 8 \cdot \frac{(E_2 - E_1)}{(E_3 - E_1)}, 0, 0, 8 \cdot \frac{(E_3 - E_2)}{(E_3 - E_1)}, 0, 0\right\}. \tag{4.5.4}$$

Looking at (4.5.3), most of the contributions come from 0. With a larger number of eigenvalues, we see that the contribution from the zero spacing of degenerate eigenvalues becomes a problem and destroys the spectral statistics. We see direct evidence of this from the spectrum of the non-magnetic and $B = 2$ magnetic Laplacian below.

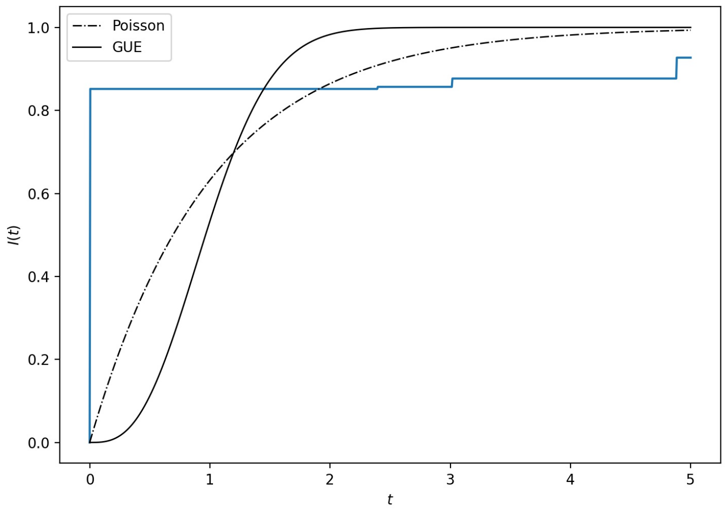

Figure 4.22: The integrated level spacing of (4.3.23c) with $N = 20$ and $k = 0$. The large contribution coming from the large numbers of 0 for the level spacing.

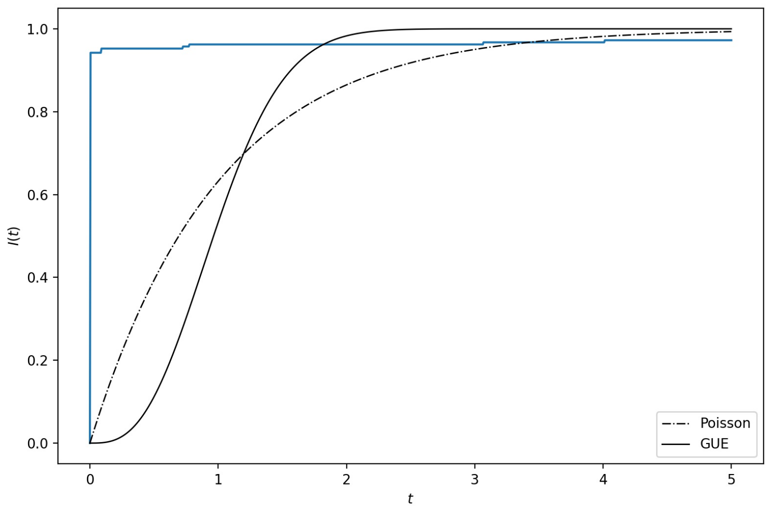

Figure 4.23: The integrated level spacing (4.3.23c) with $N = 20$ and $k = 2$. We again see the large contribution from the 0 valued level spacing.

Increasing the non-homogeneity however, destroys the degeneracy and thus will result in a better test for the spectral statistics. This can be seen in the plots below.

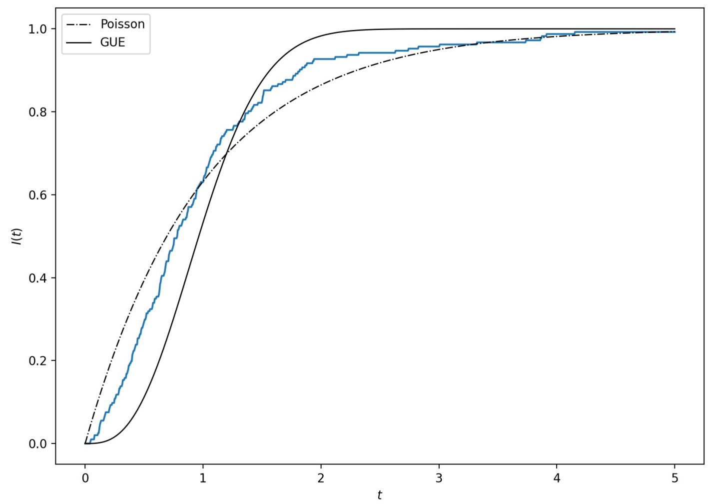

Figure 4.24: The integrated level spacing of the non-homogeneous magnetic field with $B = 2$, $N = 20$ and $\epsilon = 20$. A non-zero non-uniformity destroys the degeneracy of the spectrum yielding a more reliable test for the spectral statistics.

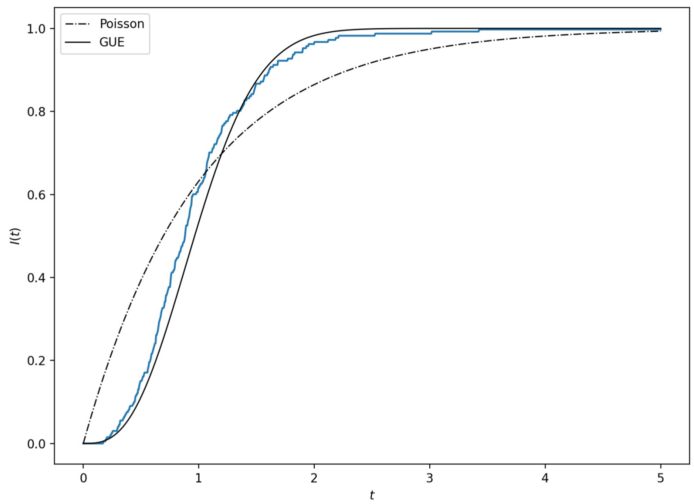

Figure 4.25: The integrated level spacing for the non-homogeneous magnetic field with $B = 2$, $N = 20$ and $\epsilon = 200$.

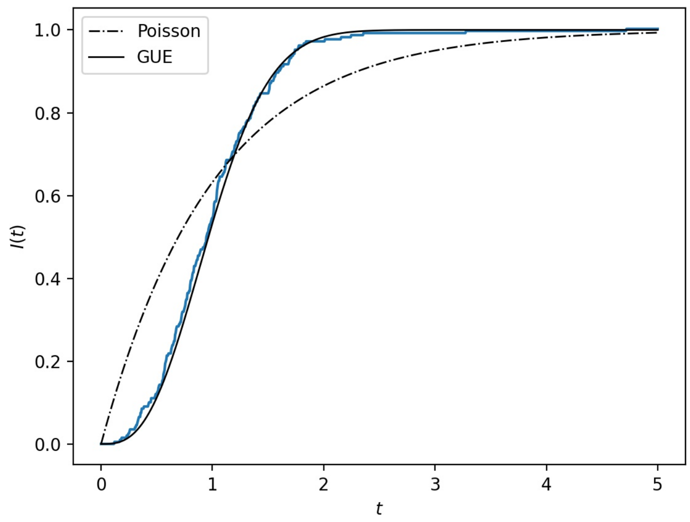

Figure 4.26: The integrated level spacing for the non-homogeneous magnetic field with $B = 2$, $N = 20$ and $\epsilon = 200,000$.

Chapter 5

Trace formulae

In this chapter we first develop a trace formula for the non-magnetic Laplacian by calculating the Weyl term and the oscillating contribution and comparing with the density of states. We then do the same but this time for the case of the Landau gauge. Then, studying the density of states by picking a suitable test function, we obtain information about the periodic orbit structure. This method is particularly useful when a Hamiltonian is not directly obtainable from a Hamiltonian matrix or if the periodic orbits of the classical system are difficult to obtain.

5.1 Background

To begin, we provide some details on the trace formula first, generally, then for integrable systems. The content in this chapter follows mostly that of [Bol97, MVB77a].

The semiclassical trace formula obtains from certain "spectral functions" information about the classical dynamics of the corresponding classical system. Most notably, it seeks to obtain correction terms to the spectral staircase and density of states. The spectral staircase is defined as

$$
\begin{aligned}
N(E) &:= \#\{n;\ E_n \le E\} && (5.1.1)\\
&= \frac{1}{(2\pi\hbar)^n}\int\!\!\int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}\,\Theta(E - H(\boldsymbol{p},\boldsymbol{q})) + \text{higher order terms}, && (5.1.2)
\end{aligned}
$$

as $\hbar \to 0$ and the density of states,

$$
\begin{aligned}
d(E) &= \frac{\mathrm{d}N(E)}{\mathrm{d}E} = \sum_n \delta(E - E_n)\\
&= \frac{1}{(2\pi\hbar)^n}\int\!\!\int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}\,\delta(E - H(\boldsymbol{p},\boldsymbol{q})) + \text{higher order terms} \quad (5.1.3)
\end{aligned}
$$

as $\hbar \to 0$.

We can obtain the spectral staircase by simply integrating the density of states, therefore, we will purely focus on the density of states. To study the corrections to the density of states the time evolution operator $\hat{U}(t) = \mathrm{e}^{-\frac{i\hat{H}t}{\hbar}}$ is considered, such that a vector $\psi_t : \mathbb{R} \times \mathbb{R}^n \to L^2(\mathbb{R}^n)$ obeys

$$
\psi_t(\boldsymbol{q}) = \left(\hat{U}(t)\psi_0\right)(\boldsymbol{q}) = \int_{\mathbb{R}^n} K(\boldsymbol{q},\boldsymbol{y};t)\psi_0(\boldsymbol{y})\mathrm{d}^n\boldsymbol{y}, \tag{5.1.4}
$$

where

$$
K(\boldsymbol{q},\boldsymbol{y};t) = \sum_j \phi(\boldsymbol{q})\overline{\phi(\boldsymbol{y})}\mathrm{e}^{-\frac{iE_jt}{\hbar}} \tag{5.1.5}
$$

and the kernel obeys the Schrödinger equation with the necessary boundary conditions.

Since $\mathrm{Tr}\hat{U}(t)$ has singularities one seeks to regularise the trace with a test function, $\rho \in \mathscr{S}(\mathbb{R}^n)$ with compact support. Using this test function we compute the trace of the evolution operator to obtain a convergent spectral function which yields

$$
\mathrm{Tr}\hat{U}\left[\frac{\hat{\rho}(t)\mathrm{e}^{\frac{iEt}{\hbar}}}{2\pi}\right] = \frac{1}{2\pi}\int_{\mathbb{R}^n}\mathrm{d}^n\boldsymbol{y}\int_{\mathbb{R}}\mathrm{d}t\,\hat{\rho}(t)\mathrm{e}^{\frac{iEt}{\hbar}}K(\boldsymbol{y},\boldsymbol{y};t) = \sum_j \rho\left(\frac{E_j - E}{\hbar}\right), \tag{5.1.6}
$$

where

$$\hat{\rho}(t) := \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-i\omega t} \rho(\omega) \mathrm{d}\omega. \tag{5.1.7}$$

The semiclassical nature of this expression comes from the semiclassical approximation to the kernel. The first approximation came from Pauli then Van-Vleck and then later for arbitrary time, Gutzwiller [Gut67, Gut69, Gut71, Gut70]. See also [JB99] where the semiclassical approximation is applied to the Dirac equation.

The semiclassical approximation to the kernel, (5.1.6), has the following form

$$K(\boldsymbol{q}, \boldsymbol{y}; t) = \frac{1}{(2\pi i\hbar)^{\frac{n}{2}}} \sum_{\boldsymbol{\gamma}} \left| \det \left(-\frac{\partial^2 S}{\partial q^i \partial y^j} \right) \right|^{\frac{1}{2}} \mathrm{e}^{\frac{i}{\hbar} S(\boldsymbol{q}, \boldsymbol{y}; t) - \frac{i\pi\mu}{2}} \{1 + \mathcal{O}(\hbar)\}, \tag{5.1.8}$$

where γ is a solution of the classical equations of motion with boundary conditions $\boldsymbol{\gamma}(0) = \boldsymbol{y}$, $\boldsymbol{\gamma}(t) = \boldsymbol{q}$, μ counts the number of conjugate points along the orbit (i.e., where $\det(\partial p_i / \partial q^j) \to \pm\infty$ if the local coordinates in phase space are $(\boldsymbol{q}, \boldsymbol{p}) = (q^1, ..., q^n, p_1, ..., p_n)$), and $S(\boldsymbol{q}, \boldsymbol{y}; t)$ the generating function which solves the HJE.

Inserting (5.1.8) into (5.1.6) we have a $\hbar$ dependence in the exponential which causes the integrand to rapidly oscillate in the semiclassical limit, causing cancellations in the integral for nearby points. We can thus use the method of stationary phase (see theorem A.0.2) to obtain an asymptotic expansion as $\hbar \to 0$. The stationary points are found to be the periodic orbits of the classical system. Since any point on the periodic orbit is a solution to the stationary phase condition, we end up with a one-dimensional manifold of stationary points. One can over come this by choosing a parameterisation of the orbit, and performing the integration over isolated orbits in the transversal coordinate. There is a vast amount of literature one can quote, see for example [Gut67, Gut69, Gut71, Gut70]. The trace formula then has the following form

$$\sum_n \rho\left(\frac{E_n - E}{\hbar}\right) \sim_{(\hbar\to 0)} \hat{\rho}(0)\overline{\rho} + \sum_\gamma \frac{A_\gamma \hat{\rho}(T_\gamma)}{2\pi} \mathrm{e}^{\frac{i}{\hbar}S_\gamma + \frac{i\pi\mu_\gamma}{2}}, \tag{5.1.9}$$

where $\overline{\rho}\hat{\rho}(0)$ is the Weyl-term, $T_{\boldsymbol{\gamma}}$ is the period of the periodic orbit labelled by $\boldsymbol{\gamma}$, $\mu_{\boldsymbol{\gamma}}$ is the number of conjugate points along the periodic orbit $\boldsymbol{\gamma}$ and $S_{\boldsymbol{\gamma}}$ is the action of the periodic orbit $\boldsymbol{\gamma}$. In later treatments of the trace formula we let $n_W(E) := \hat{\rho}(0)\overline{\rho}$ when $\rho(t) = \delta(t)$. Choosing a suitable test function the right-hand side becomes a series with peaks centred at the periodic orbits.

The trace formula allows one to obtain information about the classical periodic orbits of the system from the spectrum of the corresponding quantum Hamiltonian. Without explicit knowledge of the classical Hamiltonian one can obtain the periodic orbits of the classical system by considering a test function with compact support. The peaks in the Fourier transform are centred at the periodic orbits. This is extremely attractive for systems which exhibit purely quantum phenomena i.e., spin systems (see e.g., [JB98], [JB99]). The converse is also possible. If the quantum problem very complex, then a semiclassical approximation can provide information about the spectral density through the periodic orbits of the classical system.

5.2 Trace-Formula for integrable systems

We now give a description of a trace formula from [MVB77a] which can be used to obtain an expression for the density of states of an Liouville integrable system. In the same paper, a corresponding trace formula (found in [MVB76]) is shown to yield an equivalent expression which relies on knowledge of the energy surface. For systems of two degrees-of-freedom, this form can be useful as the stationary phase formula can be avoided when the energy surface has curvature zero.

We begin with an expression for the density of states in the form

$$d(E) = \mathrm{Tr}[\delta(E - \mathrm{Op}_\hbar^{\mathcal{A}}[h])] = \mathrm{Re}\frac{1}{\pi\hbar}\int_0^\infty \mathrm{d}t \mathrm{e}^{iEt/\hbar} \int \mathrm{d}\boldsymbol{q} K(\boldsymbol{q}, \boldsymbol{q}; t), \quad (5.2.1)$$

where K has a semiclassical approximation of the form

$$K_{\mathrm{sc}}(\boldsymbol{q}_B, \boldsymbol{q}_A; t) = \frac{1}{(2\pi i\hbar)^{n/2}} \sum_r |D_r|^{1/2} \exp\left[i\left(\frac{W_r}{\hbar} - \alpha_r\frac{\pi}{2}\right)\right], \quad (5.2.2)$$

with

$$D_r = D_r(\boldsymbol{q}_B, \boldsymbol{q}_A; t) := \det\left(-\frac{\partial^2 W_r}{\partial q_A^i \partial q_B^j}\right) = \left(\frac{\mathrm{d}\boldsymbol{q}_B}{\mathrm{d}\boldsymbol{p}_{Ar}}\right)^{-1} \quad (5.2.3)$$

and

$$W_r = W_r(\boldsymbol{q}_B, \boldsymbol{q}_A; t) := \int_0^t \mathrm{d}t' \left[\langle \boldsymbol{p}_r(t'), \dot{\boldsymbol{q}}_r(t')\rangle - H(\boldsymbol{q}_r(t'), \boldsymbol{p}_r(t'))\right], \quad (5.2.4)$$

where r labels the r-th path and $\langle\cdot,\cdot\rangle$ denoted the Euclidean inner product. Changing to action angles variables using $\boldsymbol{\theta}_B - \boldsymbol{\theta}_A = \boldsymbol{\omega}(\boldsymbol{J}_r)t$, with

$$W_r(\boldsymbol{\theta}_A, \boldsymbol{\theta}_B; t) := \langle \boldsymbol{J}_r, (\boldsymbol{\theta}_B - \boldsymbol{\theta}_A)\rangle - H(\boldsymbol{J}_r)t, \quad (5.2.5)$$

and

$$D_r(\boldsymbol{\theta}_A, \boldsymbol{\theta}_B; t) := \left(\frac{\mathrm{d}\boldsymbol{\theta}_B}{\mathrm{d}\boldsymbol{J}_r}\right)^{-1} = \frac{1}{t^n \det(\partial\boldsymbol{\omega}/\partial\boldsymbol{J}_r)}, \quad (5.2.6)$$

we obtain

$$d(E) = \frac{1}{\pi\hbar}\mathrm{Re}\frac{1}{(2\pi i\hbar)^{n/2}}\int_0^\infty \mathrm{d}t\mathrm{e}^{iEt/\hbar}\int \mathrm{d}\boldsymbol{\theta}\frac{\mathrm{e}^{\frac{i}{\hbar}\langle \boldsymbol{J}_r, (\boldsymbol{\theta}_B - \boldsymbol{\theta}_A)\rangle - \frac{i}{\hbar}H(\boldsymbol{J}_r)t - i\alpha_r\frac{\pi}{2}}}{|t^n \det(\partial\boldsymbol{\omega}/\partial\boldsymbol{J}_r)|^{1/2}}. \quad (5.2.7)$$

Due to the periodicity of the action variables the summation over all contributions, $\boldsymbol{\theta} + 2\pi\boldsymbol{M} = \boldsymbol{\theta}$, is to be be considered, this eliminates the $\boldsymbol{\theta}$ dependence and the resulting integral yields a factor of $(2\pi)^n$. The action variables, $\boldsymbol{J}_{\boldsymbol{M}}$, are taken to be those that satisfy

$$\boldsymbol{\omega}(\boldsymbol{J}_{\boldsymbol{M}}(t))t = 2\pi\boldsymbol{M}. \quad (5.2.8)$$

The r-th path has now been replaced with the specification of the topology of the orbit which are labelled by the vector $\boldsymbol{M}$.

Putting this together we obtain

$$d(E) = \frac{(2\pi)^n}{\pi\hbar}\mathrm{Re}\frac{1}{(2\pi i\hbar)^{n/2}}\sum_{\boldsymbol{M}}{}' \int_0^\infty \mathrm{d}t \frac{\mathrm{e}^{\frac{i}{\hbar}(\langle 2\pi \boldsymbol{J}_{\boldsymbol{M}}(t),\boldsymbol{M}\rangle + [E-H(\boldsymbol{J}_{\boldsymbol{M}}(t))t]) - i\frac{\pi}{2}\langle \boldsymbol{\alpha}_{\boldsymbol{M}},\boldsymbol{M}\rangle}}{|t^n \det(\partial\boldsymbol{\omega}/\partial \boldsymbol{J}_{\boldsymbol{M}}(t))|^{1/2}} + n_W(E), \tag{5.2.9}$$

where the prime on the summation indicates the exclusion of $\boldsymbol{M} = \boldsymbol{0}$ and $\langle \boldsymbol{\alpha}_{\boldsymbol{M}}, \boldsymbol{M}\rangle$ equals the total number of caustics in the whole path.

We apply the method of stationary phase with,

$$\begin{aligned}
\phi(t) &= \langle 2\pi \boldsymbol{J}_{\boldsymbol{M}}(t), \boldsymbol{M}\rangle + [E - H(\boldsymbol{J}_{\boldsymbol{M}}(t))t]\,, \\
\frac{\partial \phi(t^*)}{\partial t} &= \langle 2\pi \frac{\partial \boldsymbol{J}_{\boldsymbol{M}}(t^*)}{\partial t}, \boldsymbol{M}\rangle + (E - H(\boldsymbol{J}_{\boldsymbol{M}}(t^*)) - t\langle \frac{\partial \boldsymbol{J}_{\boldsymbol{M}}(t^*)}{\partial t}, \frac{\partial H(\boldsymbol{J}_{\boldsymbol{M}})}{\partial \boldsymbol{J}_{\boldsymbol{M}}}\rangle \\
&= \langle \frac{\partial \boldsymbol{J}_{\boldsymbol{M}}(t^*)}{\partial t}, 2\pi \boldsymbol{M} - t^*\boldsymbol{\omega}(\boldsymbol{J}_{\boldsymbol{M}})\rangle + (E - H(\boldsymbol{J}_{\boldsymbol{M}}(t^*))) \\
&= (E - H(\boldsymbol{J}_{\boldsymbol{M}}(t^*))) = 0
\end{aligned} \tag{5.2.10}$$

and

$$\phi''(t^*) := \left.\frac{\partial^2 \phi(t)}{\partial t^2}\right|_{t=t^*} = -\langle \boldsymbol{\omega}(\boldsymbol{J}_{\boldsymbol{M}}), \frac{\partial \boldsymbol{J}_{\boldsymbol{M}}(t^*)}{\partial t}\rangle, \tag{5.2.11}$$

where $\boldsymbol{J}_{\boldsymbol{M}}(t)$ is defined via (5.2.8). This yields

$$\begin{aligned}
d(E) = n_W(E) &+ \\
&+ \mathrm{Re}\sum_{\boldsymbol{M}}{}' A_{\boldsymbol{M}} \mathrm{e}^{\frac{i}{\hbar}\langle 2\pi \boldsymbol{J}_{\boldsymbol{M}}(t^*),\boldsymbol{M}\rangle - i\frac{\pi}{2}\langle \boldsymbol{\alpha}_{\boldsymbol{M}},\boldsymbol{M}\rangle + i\frac{\pi}{4}\mathrm{sign}(\phi''(t^*))} + \mathcal{O}(\hbar^{\frac{3}{2}}),
\end{aligned} \tag{5.2.12}$$

where

$$A_{\boldsymbol{M}} = \frac{2(2\pi)^{\frac{n-1}{2}}}{\hbar^{\frac{n+1}{2}} i^{\frac{n}{2}}} \frac{1}{|(t^*)^n \det(\partial\boldsymbol{\omega}/\partial \boldsymbol{J}_{\boldsymbol{M}}(t^*))|^{1/2} |\phi''(t^*)|^{1/2}}. \tag{5.2.13}$$

An expression for the density of states can thus be obtained via the Hamiltonian of a Liouville integrable system by reducing it to action-angle variables. The periodic orbits of the system are represented by the summation over $\boldsymbol{M}$ which is a vector of integers specifying the topology of the orbit.

The Berry-Tabor trace formula can be smoothed by applying a smoothing procedure to the trace formula. This results in a faster convergence of the density if states and can be seen by multiplying each term in the trace formula by $\exp(-\gamma T(\boldsymbol{M}))$, where $\gamma \ll E$ and $T(\boldsymbol{M})$ is the periodic orbit with topology $\boldsymbol{M}$, see [MVB76] for a more detailed discussion.

The density of states for an integrable system can also be written as

$$d(E) = \sum_{M_1=0}^{\infty} \dots \sum_{M_n=0}^{\infty} \delta\left(E - H\left(\left(\boldsymbol{M} + \frac{\alpha}{4}\right)\hbar\right)\right). \tag{5.2.14}$$

Using the Poisson summation formula, we let $\boldsymbol{M} \in \mathbb{Z}^n$ such that the integration can be taken over the positive quadrant and, changing variable to $\boldsymbol{I} = (\boldsymbol{M} + \boldsymbol{\alpha})\hbar$ we obtain

$$d(E) = \frac{1}{\hbar^n} \sum_{\boldsymbol{M} \in \mathbb{Z}^n} \int_{\text{+ve quadrant}} \mathrm{d}^n \boldsymbol{I} \delta(E - H(\boldsymbol{I})) \mathrm{e}^{\frac{2\pi i}{\hbar}\langle \boldsymbol{M}, \boldsymbol{I}\rangle - \frac{\pi i}{2}\langle \boldsymbol{\alpha}, \boldsymbol{M}\rangle}. \tag{5.2.15}$$

To eliminate the integral in (5.2.15) we use equation (11.5) in [FCF80], which for completeness we state here:

$$\int \mathrm{d}^n \boldsymbol{I} \delta(P) = \int_{P=0} \frac{1}{|\nabla P|} \mathrm{d}\boldsymbol{\sigma}, \tag{5.2.16}$$

where $\mathrm{d}\boldsymbol{\sigma}$ is the surface element of $P = 0$. The density of states then becomes

$$d(E) = \frac{1}{\hbar^n} \sum_{\boldsymbol{M} \in \mathbb{Z}^n} \int_{E=H(I)} \mathrm{d}\boldsymbol{\eta} \frac{\mathrm{e}^{\frac{2\pi i}{\hbar}\langle \boldsymbol{M}, \boldsymbol{I}(\boldsymbol{\eta})\rangle - \frac{\pi i \langle \boldsymbol{\alpha}, \boldsymbol{M}\rangle}{2}}}{|\boldsymbol{\omega}(\boldsymbol{I}(\boldsymbol{\eta}))|} \tag{5.2.17}$$

where the $(n-1)$-coordinates, $\boldsymbol{\eta}$, parameterise the energy shell, $H(\boldsymbol{I}) = E$.

Considering the term $\boldsymbol{M} = \mathbf{0}$, reduces (5.2.15) to the following form

$$\begin{aligned} d_{\boldsymbol{M}=\mathbf{0}}(E) &= \frac{1}{\hbar^n} \int_{+\text{ve quadrant}} \mathrm{d}^n \boldsymbol{I} \; \delta(E - H(\boldsymbol{I})) \\ &= \frac{1}{(2\pi\hbar)^n} \int \mathrm{d}\boldsymbol{\theta} \int \mathrm{d}\boldsymbol{I} \; \delta(E - H(\boldsymbol{I})) \\ &= \frac{1}{(2\pi\hbar)^n} \int \mathrm{d}\boldsymbol{q} \int \mathrm{d}\boldsymbol{p} \; \delta(E - H(\boldsymbol{q}, \boldsymbol{p})) \\ &= n_W(E), \end{aligned} \tag{5.2.18}$$

where we have used a canonical transformation to transform the action-angle variables to the original $(\boldsymbol{q}, \boldsymbol{p})$. We thus see the $\boldsymbol{M} = \mathbf{0}$ term corresponds to the Thomas-Fermi term. We are thus left with

$$d(E) = n_W(E) + \frac{1}{\hbar^n} \sum_{\boldsymbol{M} \in \mathbb{Z}^n}{}' \int_{E=H(I)} \mathrm{d}\boldsymbol{\eta} \frac{\mathrm{e}^{\frac{2\pi i}{\hbar} \langle \boldsymbol{M}, \boldsymbol{I}(\boldsymbol{\eta}) \rangle - \frac{\pi i \langle \boldsymbol{\alpha}, \boldsymbol{M} \rangle}{2}}}{|\boldsymbol{\omega}(\boldsymbol{I}(\boldsymbol{\eta}))|}. \tag{5.2.19}$$

One then proceeds to evaluate this integral with the method of stationary phase. The current form is suitable for systems where the energy surface has zero curvature, the integral can be performed exactly and so we leave it in this form.

5.3 Non-magnetic Laplacian

In this section we briefly discuss the application of the semiclassical trace formula to the case of a free particle without the presence of a magnetic field on a torus phase space.

5.3.0.1 Weyl term

The derivation of the semiclassical trace formula singles out those classical paths which start and end at the same point. Within this class of motion are two types of trajectory, those of zero length and those of non-zero length. The zero length contribution is also known as the Weyl term and is calculated by considering the total volume of the n-dimensional energy shell in phase space, divided by Planck's constant to the power of n, h^n. This has the interpretation that each quantum state occupies a volume of h^n of the classical phase space. For a two dimensional system this is written as

$$n_W(E) := \frac{\mathrm{vol}(H^{-1}(E))}{h^2} = \frac{\mathrm{vol}(H^{-1}(E))}{(2\pi\hbar)^2}. \tag{5.3.1}$$

The volume of the energy surface with the Hamilton (4.2.14) is

$$\begin{aligned}
\mathrm{vol}(H^{-1}(E)) &= \int \delta\left(E - H(\boldsymbol{p})\right) \mathrm{d}\boldsymbol{z} \\
&= \int \delta\left(E - \frac{\ell_p^2}{2\pi^2}\left(2 - \cos\left(\frac{2\pi p_1}{\ell_p}\right) - \cos\left(\frac{2\pi p_2}{\ell_p}\right)\right)\right) \mathrm{d}\boldsymbol{z} \\
&= \frac{2\pi^2}{\ell_p^2} \int \delta\left(\frac{2\pi^2 E}{\ell_p^2} - 2 + \cos\left(\frac{2\pi p_1}{\ell_p}\right) + \cos\left(\frac{2\pi p_2}{\ell_p}\right)\right) \mathrm{d}\boldsymbol{z},
\end{aligned} \tag{5.3.2}$$

where

$$\int (\cdots)\ \mathrm{d}\boldsymbol{z} = \int_0^{\ell_p} \int_0^{\ell_p} \int_0^{\ell_q} \int_0^{\ell_q} (\cdots)\ \mathrm{d}q^1 \mathrm{d}q^2 \mathrm{d}p_1 \mathrm{d}p_2. \tag{5.3.3}$$

We thus have an expression of the following form

$$\mathrm{vol}(H^{-1}(E)) = \frac{2\pi^2 \ell_q^2}{\ell_p^2} \int_0^{\ell_p} I(p_2)\ \mathrm{d}p_2, \tag{5.3.4}$$

where

$$I(p_2) = \int_0^{\ell_p} \delta\left(\epsilon(p_2) + \cos\left(\frac{2\pi}{\ell_p} p_1\right)\right)\ \mathrm{d}p_1, \tag{5.3.5}$$

and

$$\epsilon(p_2) = \frac{2\pi^2 E}{\ell_p^2} - 2 + \cos\left(\frac{2\pi}{\ell_p} p_2\right). \tag{5.3.6}$$

We solve (5.3.5) by looking for the points p_1^* such that $f(p_1^*) := \epsilon(p_2) + \cos\left(\frac{2\pi}{\ell_p} p_1^*\right) = 0$ with $p_1^* \in [0, \ell_p)$. Finding the points where $f(p_1^*) = 0$ gives

$$p_1^* = \pm \frac{\ell_p}{2\pi} \arccos(\epsilon) + \ell_p m \text{ where } m \in \mathbb{Z}. \tag{5.3.7}$$

The expression $f(p_1^*) = 0$, such that $p_1 \in [0, \ell_p)$ has two solutions $p_1^* = \frac{\ell_p}{2\pi} \arccos(\epsilon(p_2)), \ell_p - \frac{\ell_p}{2\pi} \arccos(\epsilon(p_2))$ where there are no solutions for $|\epsilon(p_2)| >$

1. With this condition and using $f'(p_1) = -\frac{2\pi}{\ell_p}\sin\left(\frac{2\pi}{\ell_p}p_1\right)$ we therefore obtain

$$I = \frac{\ell_p}{\pi}\frac{1}{\sqrt{1-\epsilon(p_2)^2}} \tag{5.3.8}$$

for $|\epsilon(p_2)| < 1$ and $I = 0$ otherwise. Putting this altogether with (5.3.4), we have

$$\mathrm{vol}(H^{-1}(E)) = \frac{2\pi\ell_q^2}{\ell_p}\int_\lambda \frac{\mathrm{d}p_2}{\sqrt{1-\epsilon(p_2)^2}}, \tag{5.3.9}$$

where $\lambda := \{p_2 \in [0,\ell_p); -1 < 2 - \frac{2\pi^2 E}{\ell_p^2} - \cos\left(\frac{2\pi}{\ell_p}p_2\right) < 1\}$. The last condition separates the domain of integration in the following way: the condition

$$2 - \frac{2\pi^2 E}{\ell_p^2} - \cos\left(\frac{2\pi}{\ell_p}p_2\right) < 1$$

is satisfied by

$$p_2 \in \lambda_1(E) := \left[0, \frac{\ell_p}{2\pi}\arccos\left(1 - \frac{2\pi^2 E}{\ell_p^2}\right)\right] \cup \left[\ell_p - \frac{\ell_p}{2\pi}\arccos\left(1 - \frac{2\pi^2 E}{\ell_p^2}\right), \ell_p\right].$$

The allowed energies of this first interval are $E_2 \in (0, \frac{\ell_p^2}{\pi^2})$. The energies in the set

$$\mathcal{E} = \left\{x \middle| x \in \left(0, \frac{2\ell_p^2}{\pi^2}\right] \setminus \left[0, \frac{\ell_p^2}{\pi^2}\right]\right\}$$

give complex values to λ_1. We therefore find for $E \in (0, \frac{\ell_p^2}{\pi^2})$,

$$\mathrm{vol}(H^{-1}(E)) = \frac{4\pi\ell_q^2}{\ell_p}\int_0^{\frac{\ell_p}{2\pi}\arccos\left(1-\frac{2\pi^2 E}{\ell_p^2}\right)} \frac{\mathrm{d}p_2}{\sqrt{1-\epsilon(p_2)^2}}. \tag{5.3.10}$$

The same reasoning can be said for the condition $-1 < 2 - \frac{2\pi^2 E}{\ell_p^2} - \cos\left(\frac{2\pi}{\ell_p}p_2\right)$. The values of p_2 which satisfy this are contained in the set

$$\lambda_2(E) := \left[\frac{\ell_p}{2\pi}\arccos\left(3 - \frac{2\pi^2 E}{\ell_p^2}\right), \ell_p - \frac{\ell_p}{2\pi}\arccos\left(3 - \frac{2\pi^2 E}{\ell_p^2}\right)\right].$$

We therefore find that when $E \in (\frac{\ell_p^2}{\pi^2}, \frac{2\ell_p^2}{\pi^2})$,

$$\mathrm{vol}(H^{-1}(E)) = \frac{2\pi\ell_q^2}{\ell_p}\int_{\frac{\ell_p}{2\pi}\arccos\left(3-\frac{2\pi^2 E}{\ell_p^2}\right)}^{\ell_p - \frac{\ell_p}{2\pi}\arccos\left(3-\frac{2\pi^2 E}{\ell_p^2}\right)} \frac{\mathrm{d}p_2}{\sqrt{1-\epsilon(p_2)^2}}. \tag{5.3.11}$$

Putting this altogether we find for $E \in (0, \frac{\ell_p^2}{\pi^2})$:

$$n_W(E) = \frac{\mathrm{vol}(H^{-1}(E))}{(2\pi\hbar)^2} = \frac{4\pi N^2}{\ell_p} \int_0^{\frac{\ell_p}{2\pi}\arccos\left(1-\frac{2\pi^2 E}{\ell_p^2}\right)} \frac{\mathrm{d}p_2}{\sqrt{1-\epsilon(p_2)^2}} \tag{5.3.12}$$

and for $E \in (\frac{\ell_p^2}{\pi^2}, \frac{\ell_p^2}{\pi^2})$:

$$n_W(E) = \frac{2\pi N^2}{\ell_p^3} \int_{\frac{\ell_p}{2\pi}\arccos\left(3-\frac{2\pi^2 E}{\ell_p^2}\right)}^{\ell_p-\frac{\ell_p}{2\pi}\arccos\left(3-\frac{2\pi^2 E}{\ell_p^2}\right)} \frac{\mathrm{d}p_2}{\sqrt{1-\epsilon(p_2)^2}}, \tag{5.3.13}$$

where we have used $2\pi\hbar N = \ell_q \ell_p$. In figure 5.1 shows a plot of the Weyl term for the case of a free particle without the presence of a magnetic field in both energy intervals.

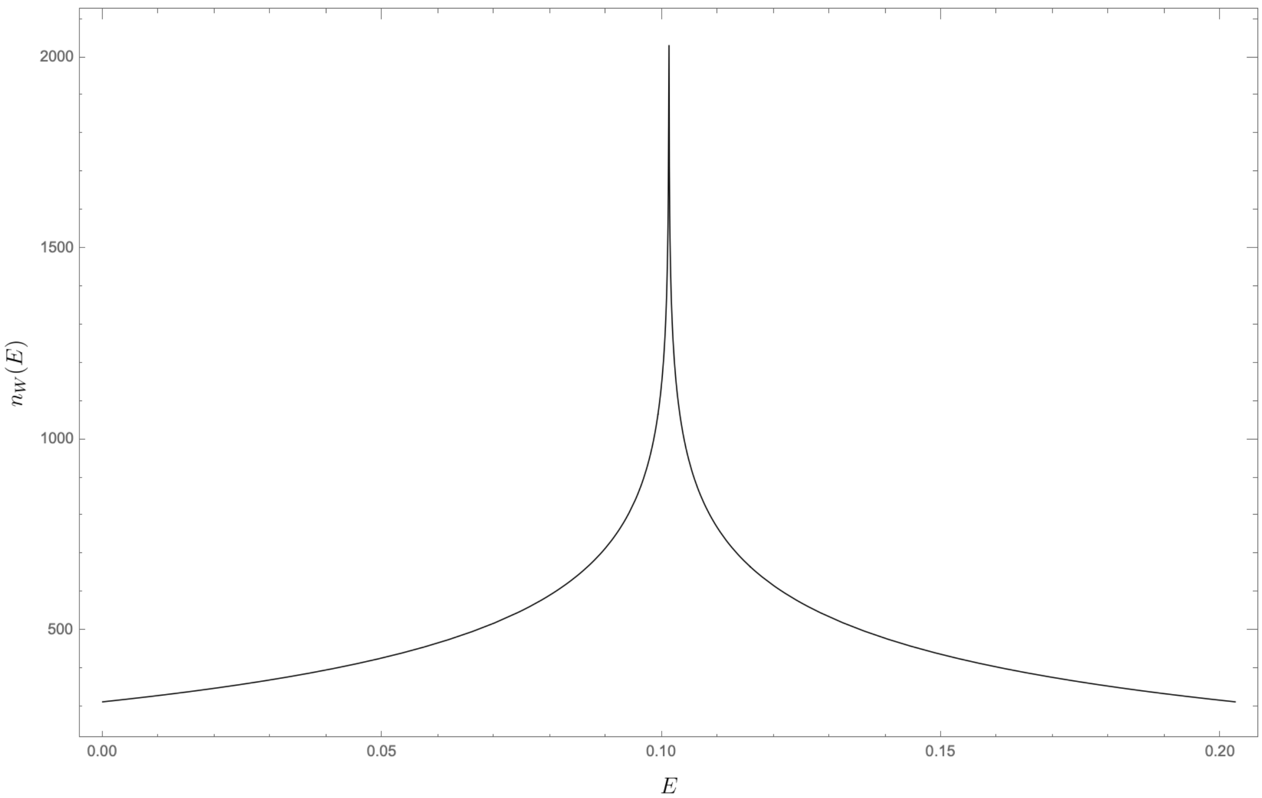

Figure 5.1: The Weyl term for the case of a free particle on a torus phase space without the presence of a magnetic field, with $N = 10$, and $\ell_p = 1$.

We notice that at $E = \ell_p^2/\pi^2$ the presence of a separatrix causes the Weyl term to diverge. At the separatrix it takes an infinite amount of time to return to a fixed point and thus the volume of the energy shell tends to infinity.

We also show a comparison of the energy eigenvalues of (4.3.23c) when $k = 0$ and the Weyl term.

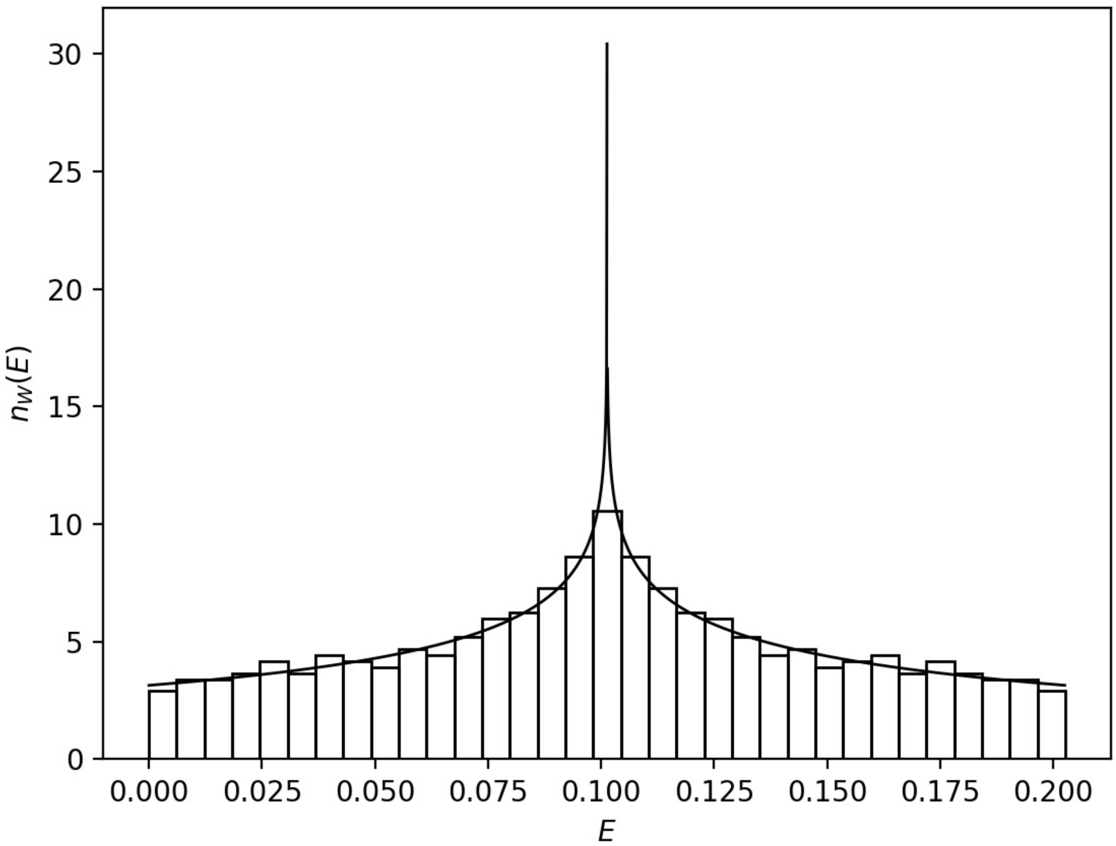

Figure 5.2: The Weyl term eigenvalues of (4.3.23c), with $N = 31$, $k = 0$ and $\ell_p = 1$. Note the area beneath the histogram and the Weyl term is normalised to one.

5.3.0.2 Oscillating contribution to the density of states

To obtain the oscillatory part of the density of states we use (5.2.12) with the Hamiltonian (4.3.28). The action variables $\boldsymbol{J_M}(t)$ which correspond to the orbit with topology $\boldsymbol{M}$ is obtained by solving (5.2.8) which gives

$$J_j^{\boldsymbol{M}}(t) = \frac{(-1)^{k_j}\ell_q\ell_p}{4\pi^2}\arcsin\left(\frac{\ell_q\pi M_j}{\ell_p t}\right) + \frac{\ell_p\ell_q k_j}{4\pi}. \tag{5.3.14}$$

We pick $k_j = 1$ since this gives a positive values for $J_j^{\boldsymbol{M}}(t)$. Calculating the other necessary quantities we have

$$\begin{aligned}\det\left(\frac{\partial\omega^i}{\partial J_j^{\boldsymbol{M}}(t)}\right) &= \left(\frac{8\pi^2}{\ell_q^2}\right)^2\sqrt{1-\left(\frac{\ell_q\pi M_1}{\ell_p t}\right)^2}\sqrt{1-\left(\frac{\ell_q\pi M_2}{\ell_p t}\right)^2}\\ &= D(\boldsymbol{M}, E),\end{aligned} \tag{5.3.15}$$

$$\begin{aligned}\langle\boldsymbol{\omega}(\boldsymbol{J}^{\boldsymbol{M}}(t)), \frac{\partial \boldsymbol{J}^{\boldsymbol{M}}(t)}{\partial t}\rangle &= \frac{\ell_q^2 M_1^2}{2t^3\sqrt{1-\left(\frac{\ell_q\pi M_1}{\ell_p t}\right)^2}} + \frac{\ell_q^2 M_2^2}{2t^3\sqrt{1-\left(\frac{\ell_q\pi M_2}{\ell_p t}\right)^2}} && (5.3.16)\\ &= G(\boldsymbol{M}, E). && (5.3.17)\end{aligned}$$

To find $t_{\boldsymbol{M}}(E)$ we substitute (5.3.14) into the Hamiltonian (4.2.14) and solve for t which yields

$$t_{\boldsymbol{M}}(E) = \sqrt{\frac{2\alpha^2\left(\frac{\ell_q\pi}{\ell_p}\right)^2(M_1^2+M_2^2)+\sqrt{R_{\boldsymbol{M}}(E)}}{2(4\alpha^2-\alpha^4)}}, \tag{5.3.18}$$

where

$$R_{\boldsymbol{M}}(E) := 4\alpha^4\left(\frac{\ell_q\pi}{\ell_p}\right)^4(M_1^2+M_2^2)^2 + 4(4\alpha^2-\alpha^4)\left(\frac{\ell_q\pi}{\ell_p}\right)^4(M_1-M_2^2)^2 \tag{5.3.19}$$

and

$$\alpha = \frac{2\pi^2 E}{\ell_p^2} - 2. \tag{5.3.20}$$

Putting this altogether we have for the oscillating contribution

$$d_{\mathrm{osc}}(E) = 2\,\mathrm{Re}\sum_{M_1,M_2=1}^{\infty} a_{\boldsymbol{M}}(E)\mathrm{e}^{\frac{2\pi i}{\hbar}\langle\boldsymbol{M},\boldsymbol{J}_1^{\boldsymbol{M}}(t_{\boldsymbol{M}}(E))\rangle} + \mathcal{O}(\hbar^{\frac{3}{2}}) \tag{5.3.21}$$

where

$$a_{\boldsymbol{M}}(E) = \sqrt{\frac{2\pi}{\hbar^3 i^3 t_{\boldsymbol{M}}(E)^2|D(\boldsymbol{M},E)|G(\boldsymbol{M},E)}}. \tag{5.3.22}$$

The oscillating part of the density of states is plotted below.

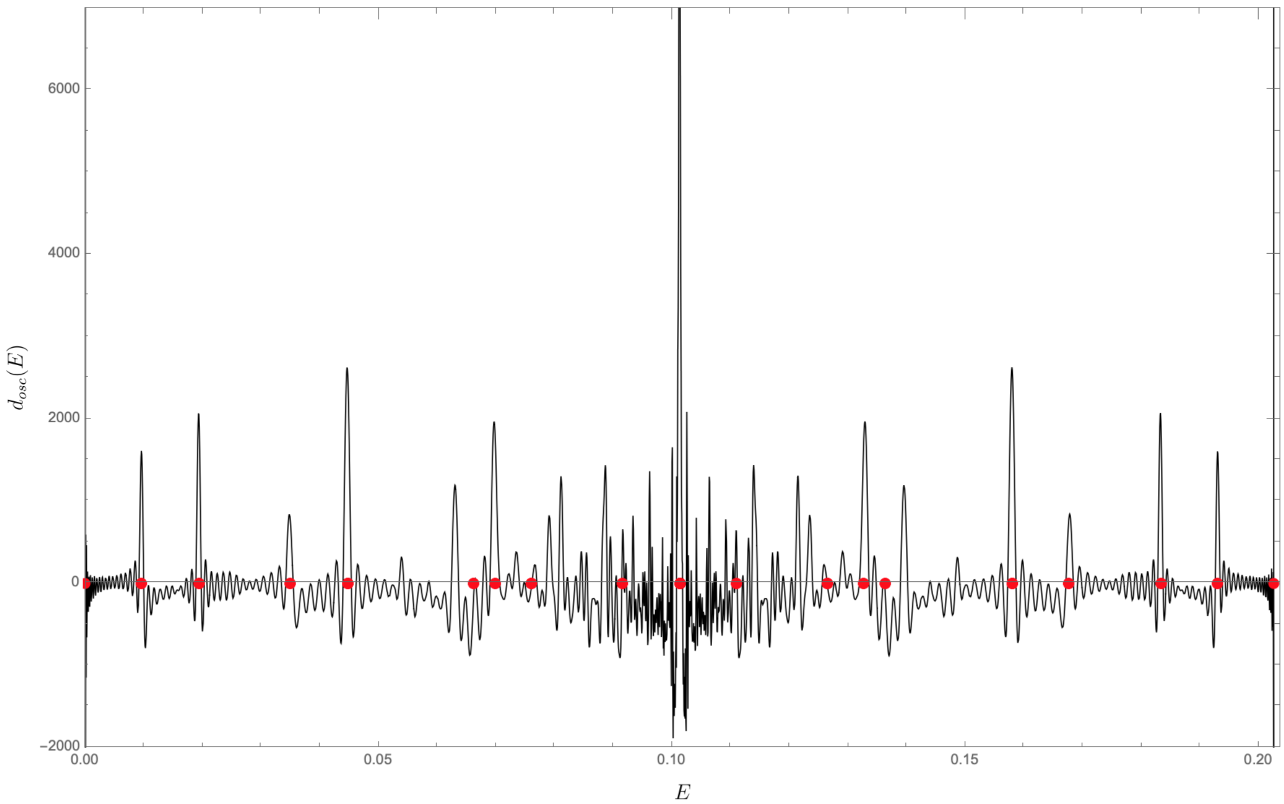

Figure 5.3: The oscillating contribution to the density of states, when $N = 10$, $\ell_q = \ell_p = 1$ with eigenvalues (4.3.23c) for $k = 0$.

Combining the Weyl term and the oscillating part of the density of states we obtain the density of states as the plot below illustrates.

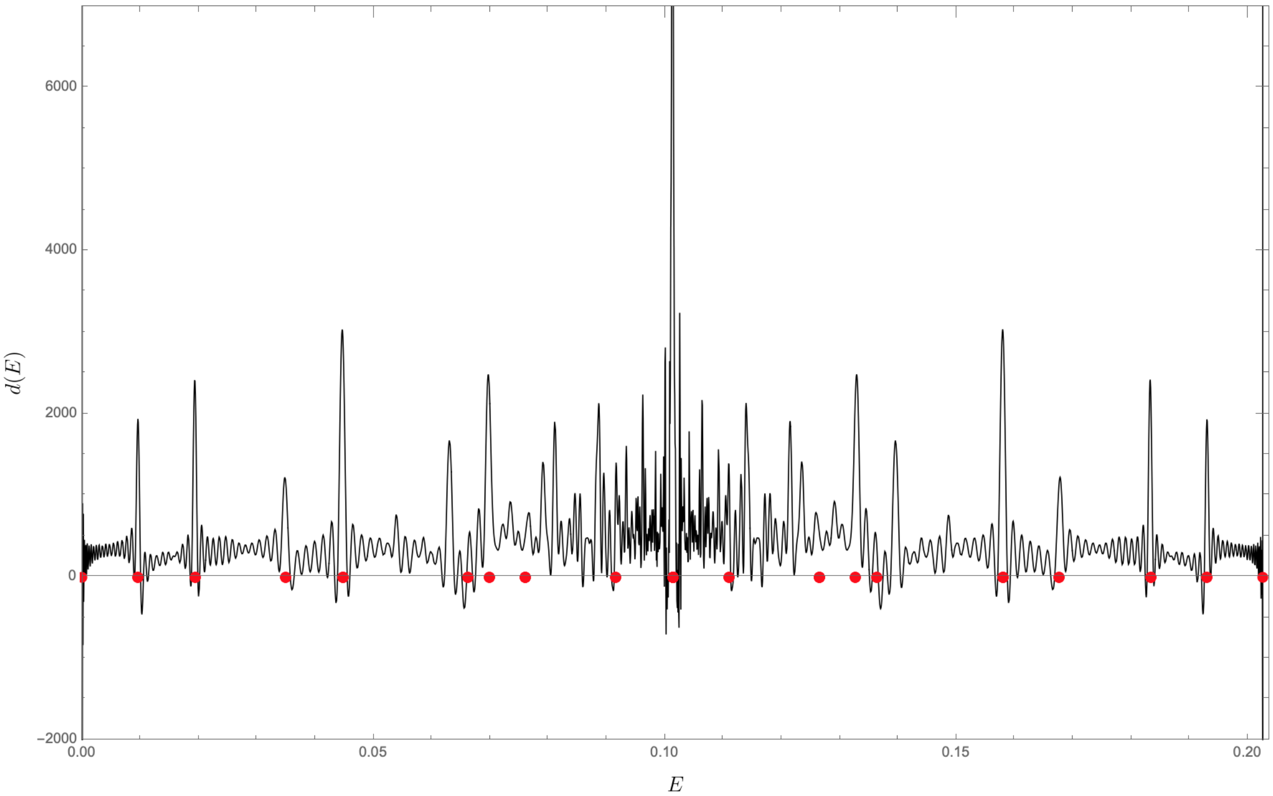

Figure 5.4: The density of states, when $N = 10$, $\ell_q = \ell_p = 1$ and the maximum value of M_1 and M_2 equals 20 with eigenvalues (4.3.23c) for $k = 0$.

As is seen in the figure the Weyl term smooths the final form, where the peaks correspond to the quantum energy levels. We now compare this to a Lorentzian smoothed density of states

$$\rho_\gamma(E) = \sum_{n=1}^{N^n} \delta_\gamma(E - E_n) = \frac{\gamma}{\pi} \sum_{n=1}^{N^n} \frac{1}{(E - E_n)^2 + \gamma^2}. \tag{5.3.23}$$

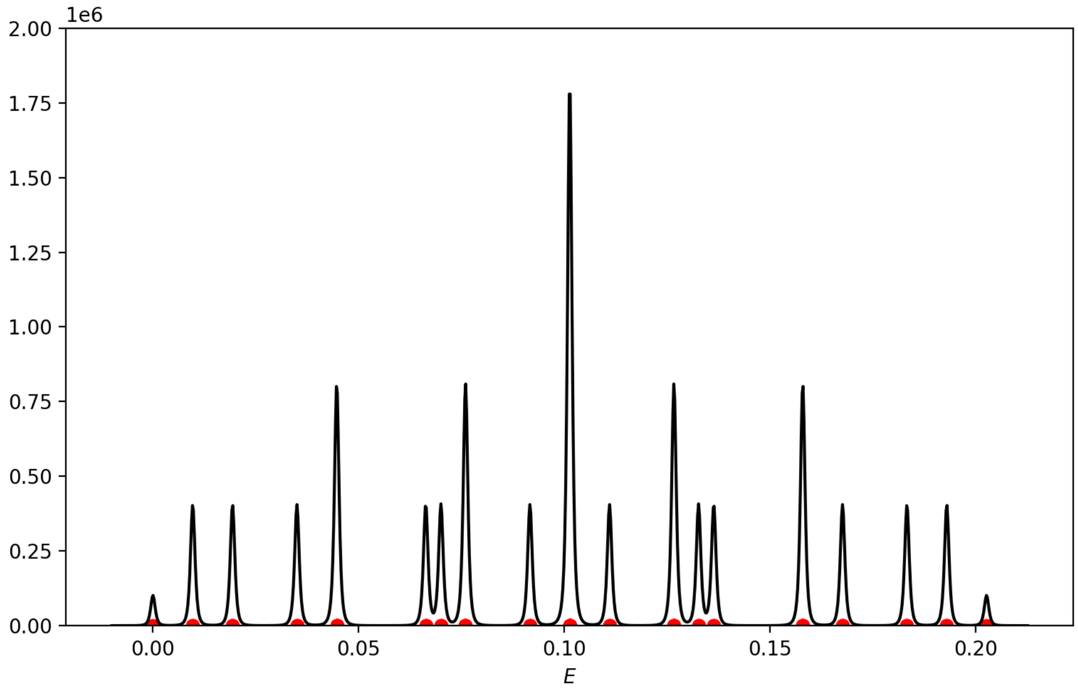

Figure 5.5: The Lorentzian smoothed density of states (5.3.23), when $N = 10$ and $\gamma = 0.001$ with the red dots being the eigenvalues (4.3.23c) for $k = 0$.

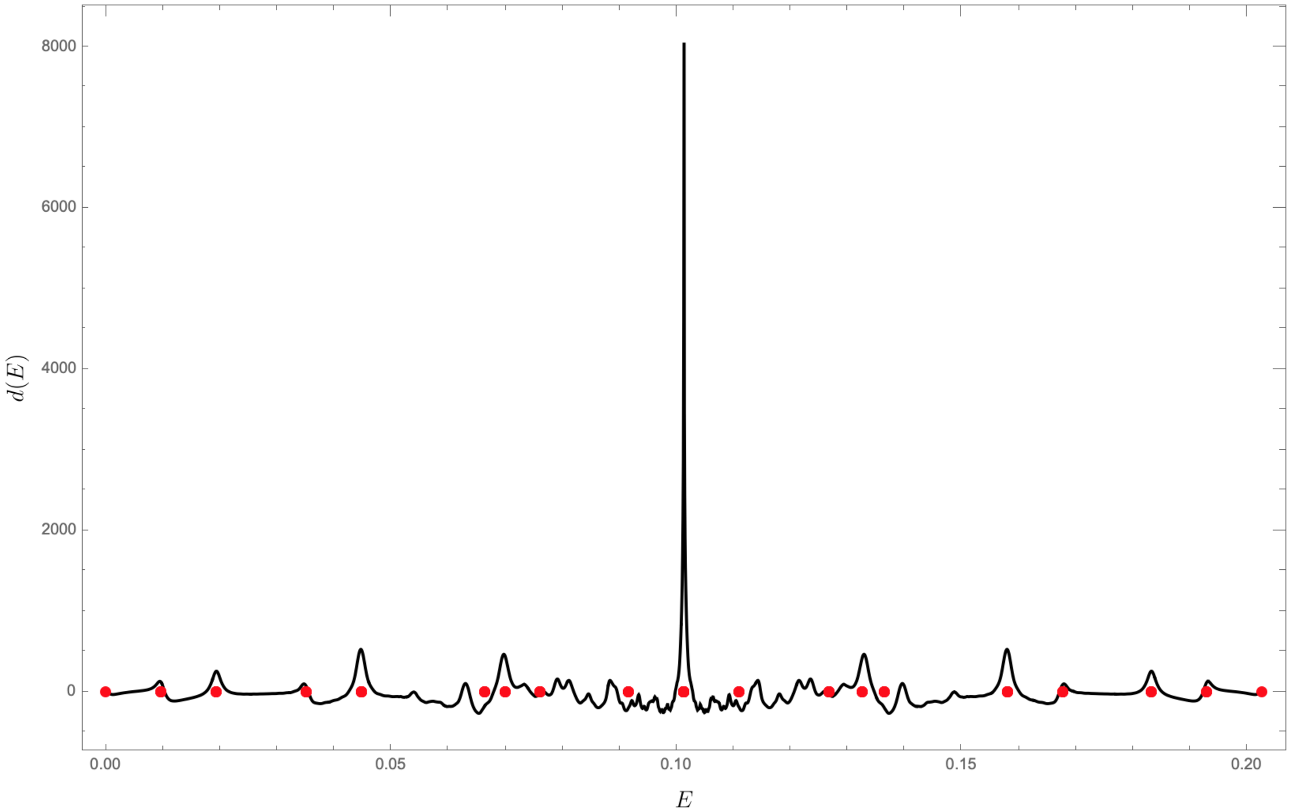

Figure 5.6: The Berry-Tabor trace smoothed by a Lorentzian factor (5.3.23), when $N = 10$ and $\gamma = 0.001$ and the maximum values M_1 and M_2 take is 20. The red dots are the eigenvalues (4.3.23c) for $k = 2$.

The peaks from the smoothed density of states and the smoothed Berry-Tabor trace formula match for the first and last segment of the energy interval. Near the separatrix the trace formula looses the ability to obtain the correct energy levels. This is because the periods of the orbits tend to infinity as they approach the separatrix and thus don't represent closed orbits. The periodic orbit expansion thus breaks down.

5.4 Trace-formula for Landau Hamiltonian

In the Landau gauge the Hamiltonian on the phase space $\mathbb{T}^4$ has the following form

$$h(\boldsymbol{q},\boldsymbol{p}) = \frac{\ell_p^2}{2\pi^2}\left(2-\cos\left(\frac{2\pi}{\ell_p}p_1+\frac{2\pi k}{\ell_q}q^2\right)-\cos\left(\frac{2\pi p_2}{\ell_p}\right)\right). \tag{5.4.1}$$

We are now in a position to use the Hamiltonian to obtain a trace formula using action angle variables. We first calculate the Weyl term and then use the above sections on the theory proposed by Berry-Tabor trace to obtain the trace formula.

5.4.1 Weyl term

The first term one wishes to calculate is the Weyl term, (5.2.18) when $\boldsymbol{M}=\boldsymbol{0}$. Using (5.4.1) in (5.2.18) we find using for the fundamental domain $\mathbb{F} := [0,\ell_q)^2 \times [0,\ell_p)^2$ that

$$\begin{aligned} n_W(E) = \frac{\ell_q}{2\hbar^2\ell_p^2}\int_0^{\ell_p}\mathrm{d}p_1\int_0^{\ell_p}\mathrm{d}p_2\int_0^{\ell_q}\mathrm{d}q^2\cdot \\ \cdot\,\delta\left(2-\frac{2\pi^2E}{\ell_p^2}-\cos\left(\frac{2\pi}{\ell_p}p_1+\frac{2\pi k}{\ell_q}q^2\right)-\cos\left(\frac{2\pi p_2}{\ell_p}\right)\right). \end{aligned} \tag{5.4.2}$$

To solve this we use a similar method as before. Letting,

$$\epsilon = 2-\frac{2\pi^2E}{\ell_p^2}-\cos\left(\frac{2\pi p_1}{\ell_p}+\frac{2\pi k q^2}{\ell_q}\right) \tag{5.4.3}$$

reduces the integral to

$$I=\int_0^{\ell_p}\mathrm{d}p_2\,\delta\left(\epsilon-\cos\left(\frac{2\pi p_2}{\ell_p}\right)\right). \tag{5.4.4}$$

Again, taking into account where the argument of the delta function is zero and evaluating the derivative of the argument at these points the integral becomes,

$$\begin{aligned} I &= \frac{\ell_p}{2\pi}\int_0^{\ell_p} \frac{\delta\left(p_2 - \frac{\ell_p}{2\pi}\arccos(\epsilon)\right)}{\sqrt{1-\epsilon^2}}\,\mathrm{d}p_2 \\ &+ \frac{\ell_p}{2\pi}\int_0^{\ell_p} \frac{\delta\left(p_2 + \frac{\ell_p}{2\pi}\arccos(\epsilon) - \ell_p\right)}{\sqrt{1-\epsilon^2}}\,\mathrm{d}p_2 \\ &= \frac{\ell_p}{\pi}\frac{1}{\sqrt{1-\epsilon^2}}. \end{aligned} \tag{5.4.5}$$

Solving the remaining integral yields,

$$n_W(E) = \frac{2\pi\ell_q^2}{\ell_p}\int_\lambda \frac{\mathrm{d}p_2}{\sqrt{1-\epsilon(p_2)^2}}, \tag{5.4.6}$$

where $\lambda := \{p_2 \in [0, \ell_p); -1 < 2 - \frac{2\pi^2 E}{\ell_p^2} - \cos\left(\frac{2\pi}{\ell_p}p_2\right) < 1\}$. The Weyl term is thus independent of the magnetic field strength and is equal to that of the case with $k = 0$. This is not surprising as the leading order expansion of the density of states does not necessarily depend on the magnetic field strength. For example, the heat kernel of the magnetic Schrödinger operator in the case of a constant magnetic field is given by the Mehler kernel, [JB13], for which the leading order of the trace does not depend on the magnetic field strength, B. The Weyl term (also know as the leading term for the average density of states) for billiards is then $A/4\pi$, where A is the area of the billiard, as is known for billiards without a magnetic field [ABS11].

Figure 5.7 shows the comparison of the Weyl term with the histogram of the eigenvalues of (4.3.23c) when $k = 2$. Due to the degeneracy, the histogram does not fit "neatly" underneath the Weyl term. This raises the question, how good of an average is the Weyl term for the periodic Landau Hamiltonian?

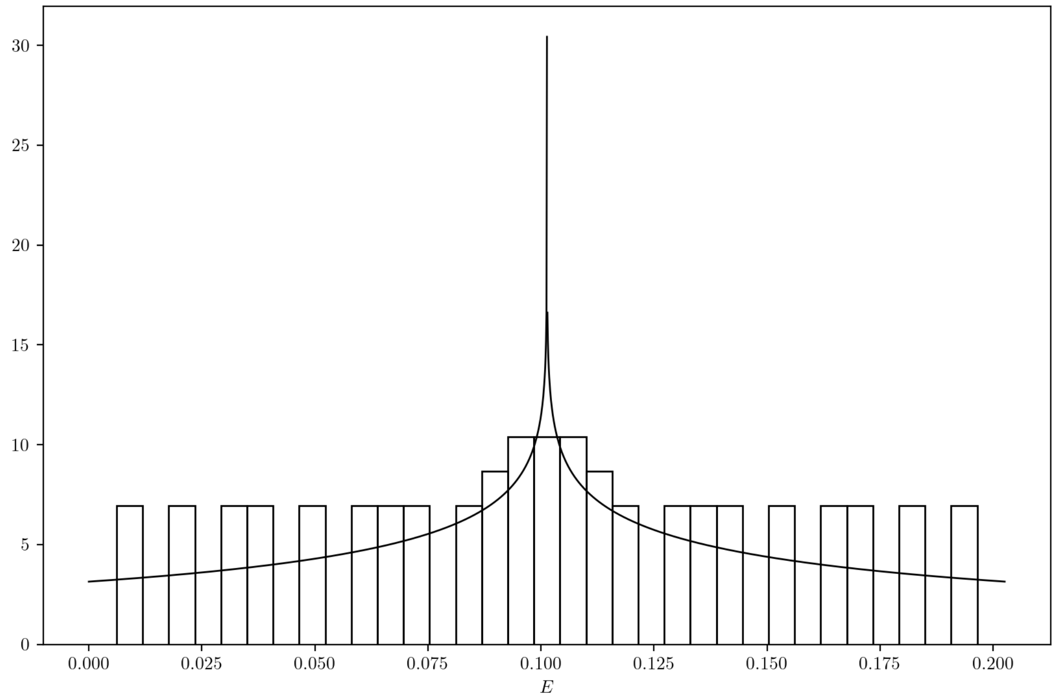

Figure 5.7: We have a histogram for the spectrum of (4.3.23c) for $N = 50$ with the Weyl term (5.4.6) which we have labelled in the plot as n_W. Note the histogram and the Weyl term has been normalised to have unit area.

We answer answer this question by plotting the counting function against the integrated Weyl term. The counting function is defined as

$$N(E) := \#\{n;\ E_n \leq E\}, \tag{5.4.7}$$

where E_n is the ordered eigenvalues of the Hamiltonian. When the Weyl term is "good" the plot should yield a straight line. When $k = 2$ the plot has a sawtooth shape, which suggests the Weyl term for the Landau Hamiltonian is a "poor" average. The plot for $k = 0$ is in closer agreement as can be seen from figure 5.2.

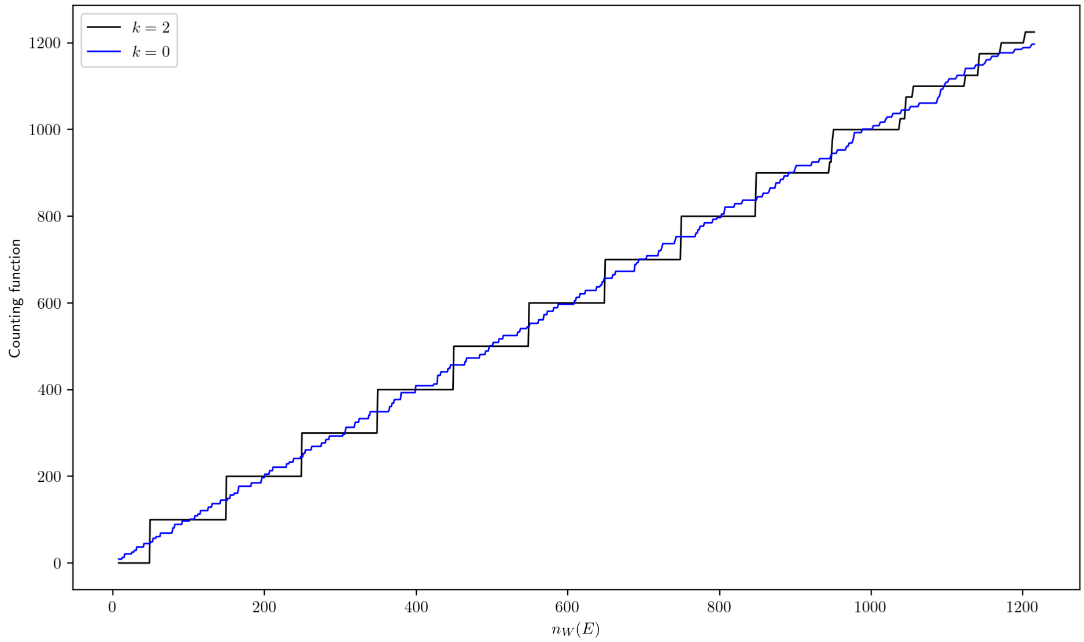

Figure 5.8: Counting function of eigenvalues vs the Weyl term (5.4.6). When $k = 2$ the plot has a sawtooth like shape, which indicates the Weyl term is a "poor" average.

5.4.2 Oscillating contribution to the trace formula

We simplify notation here slightly by letting $p_1 = \kappa \ell_p$, the q^1-momentum is now parameterised by $\kappa \in [0, 1)$.

To calculate the action variables, we notice that a change in κ shifts the orbits, this is seen by $\kappa \to \kappa + 1$ in the expression for

$$p_2(\kappa) = \pm \frac{\ell_p}{2\pi} \arccos\left(2 - 2\alpha - \cos\left(2\pi\kappa + 2\pi + \frac{2\pi k q^2}{\ell_q}\right)\right) + \ell_p m. \tag{5.4.8}$$

When $\kappa = 0$ and $k = 2$ there are two orbits where one of the orbits is centred at the origin and one at $q^2 = \ell_q/2$. Shifting κ by one, we can centre the second orbit at the origin and the first orbit now has centre $\ell_q/2$. The two orbits are therefore the same, i.e., $p_2(\kappa) = p_2(\kappa + 1)$. We showed earlier that there are k orbits. Therefore shifting κ by k, centres the

k-th orbit. Therefore all k orbits are the same, and thus the area enclosed by each orbit is the same. Since all orbits enclose the same area we can pick one for the global action variable.

The conserved quantities for the Landau Hamiltonian are $f_2 = p_1$ and $f_1 = E$. Therefore the motion takes place on the Liouville-Arnold torus $\Xi_{LAT} := \{(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{T}^4; H(\boldsymbol{q}, \boldsymbol{p}) = E, p_1 = \kappa \ell_p\}$.

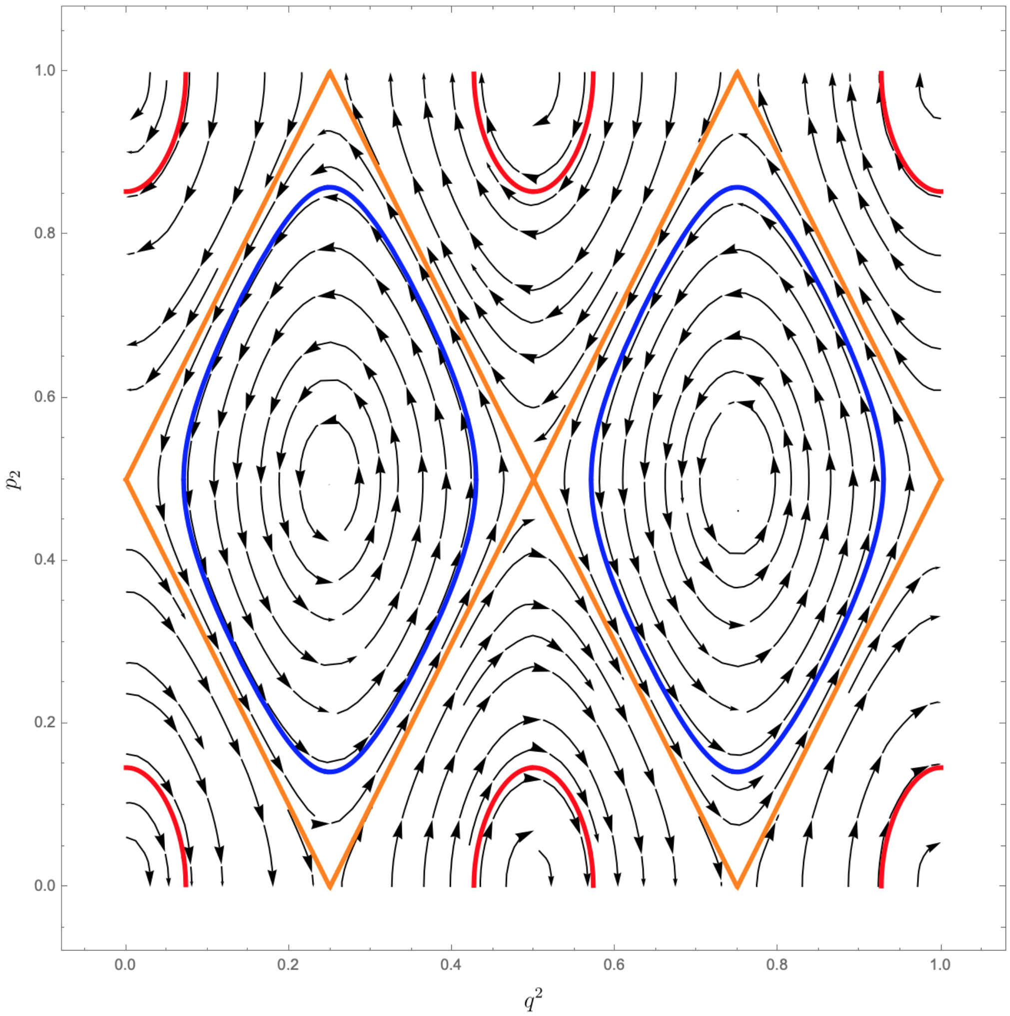

Figure 5.9: This graph shows the phase space plot for the Hamiltonian (5.4.1) in the (q^2, p_2)-plane as well as three trajectories where the blue trajectory is at $E = 0.12$, the orange trajectory is the separatrix and the red trajectory is at $E = 0.02$ with the parameter values $k = 2$, $\kappa_1 = 0$, $\ell_q = \ell_p = 1$.

Looking at figure 5.9 which contains the phase space plot in the (q^2, p_2)-plane, we find that at $E = \ell_p^2/\pi^2$ a separatrix occurs and thus the plane splits into two regions. We start first with the first energy interval $E \in (0, \frac{\ell_p^2}{\pi^2})$.

5.4.3 $E \in \left(0, \frac{\ell_p^2}{\pi^2}\right)$

To define the non-contractable loops it is seen by observation that the J_2 the second constant of motion. Since p_1 is conserved the motion will take place on a straight line with periodic boundary conditions. The first action variable is therefore defined as

$$J_1 = \frac{1}{2\pi} \oint p_1 \, \mathrm{d}q^1 = \frac{1}{2\pi} \int_0^{\ell_q} \kappa \ell_p \, \mathrm{d}q^1 = \frac{\kappa \ell_q \ell_p}{2\pi}. \tag{5.4.9}$$

Solving (5.4.1) for q^2 yields an expression for p_2. The principal branch of arccos has values in the interval $[0, \pi]$. To obtain $p_2 \in [0, \ell_p)$ we also consider the second branch,

$$p_2^r(q^2, \kappa) = \frac{\ell_p(-1)^r}{2\pi} \arccos\left(2 - \frac{2\pi^2 E}{\ell_p^2} - \cos\left(\frac{2\pi k q^2}{\ell_q} + 2\pi\kappa\right)\right) + \ell_p r. \tag{5.4.10}$$

The motion first starts on the $r = 0$ branch; when the boundary is reached the trajectory continues to the branch where $r = 1$ as the red plot shows in figure 5.9.

To find the area enclosed by an orbit we first obtain the turning points which are found by solving $p_2^0(q_i^m, \kappa) = 0$. The solution is satisfied by the following two expressions

$$\begin{aligned} q_1^m &= -\frac{\ell_q}{2\pi k} \arccos\left(1 - \frac{2\pi^2 E}{\ell_p^2}\right) - \frac{\ell_q \kappa}{k} + \frac{\ell_q m}{k}, \\ q_2^m &= \frac{\ell_q}{2\pi k} \arccos\left(1 - \frac{2\pi^2 E}{\ell_p^2}\right) - \frac{\ell_q \kappa}{k} + \frac{\ell_q m}{k}. \end{aligned} \tag{5.4.11}$$

We see that changing κ only shift the orbits and does not change the area, therefore the action variable J_2 does not change when J_1 is changed. This implies that the energy is a function of one action variable only. When the system has 2 degrees-of-freedom, the Hamiltonian can be written in the form $H(J_1, J_2) = E$, where changing one of the action variables while keeping the other fixed changes the energy of the system. Therefore, if the energy depends on one action variable, changing the other will have no effect on the energy, this implies that the energy contour has zero curvature when plotted in the (J_1, J_2)-plane.

To calculate the area enclosed, we note that for this energy range, the orbit is clockwise and thus has a positive contribution. We take as the left turning point

$$\begin{aligned} a &= -\frac{\ell_q}{2\pi k}\arccos\left(1-\frac{2\pi^2 E}{\ell_p^2}\right) - \frac{\ell_q\kappa}{k\ell_p} + \frac{\ell_q}{k} &(5.4.12)\\ &= -\frac{\ell_q\alpha}{2\pi k} - \frac{\ell_q\kappa}{k\ell_p} + \frac{\ell_q}{k}, &(5.4.13)\end{aligned}$$

and for the right turning point

$$\begin{aligned} b &= \frac{\ell_q}{2\pi k}\arccos\left(1-\frac{2\pi^2 E}{\ell_p^2}\right) - \frac{\ell_q\kappa}{k\ell_p} + \frac{\ell_q}{k} &(5.4.14)\\ &= \frac{\ell_q\alpha}{2\pi k} - \frac{\ell_q\kappa}{k\ell_p} + \frac{\ell_q}{k}, &(5.4.15)\end{aligned}$$

where $\alpha = \arccos\left(1-\frac{2\pi^2 E}{\ell_p^2}\right)$. From this we find the area enclosed and thus the action variable to be

$$\begin{aligned} J_2 &= \int_a^b (p_2 - 0)\,\mathrm{d}q^2 + \int_a^b \ell_p - (\ell_p - p_2)\,\mathrm{d}q^2\\ &= 2\int_a^b p_2\,\mathrm{d}q^2\\ &= \frac{\ell_p}{\pi}\int_{-\frac{\ell_q\alpha}{2\pi k}-\frac{\ell_q\kappa}{k\ell_p}+\frac{\ell_q}{k}}^{\frac{\ell_q\alpha}{2\pi k}-\frac{\ell_q\kappa}{k\ell_p}+\frac{\ell_q}{k}} \arccos\left(2-\frac{2\pi^2 E}{\ell_p^2} - \cos\left(\frac{2\pi k}{\ell_q}q^2 + 2\pi\kappa\right)\right)\,\mathrm{d}q^2. \end{aligned} \tag{5.4.16}$$

Changing variables and noting that the integration is of an even function over a symmetric interval we thus acquire

$$
\begin{aligned}
J_2 &= \frac{\ell_p \ell_q}{\pi^2 k} \int_0^{\alpha} \arccos\left(2 - \frac{2\pi^2 E}{\ell_p^2} - \cos\left(q^2\right)\right) \mathrm{d}q^2 \\
&= \frac{\ell_p \ell_q}{\pi^2 k} \int_0^{\arccos\left(1 - \frac{2\pi^2 E}{\ell_p^2}\right)} \arccos\left(2 - \frac{2\pi^2 E}{\ell_p^2} - \cos\left(q^2\right)\right) \mathrm{d}q^2. \qquad (5.4.17)
\end{aligned}
$$

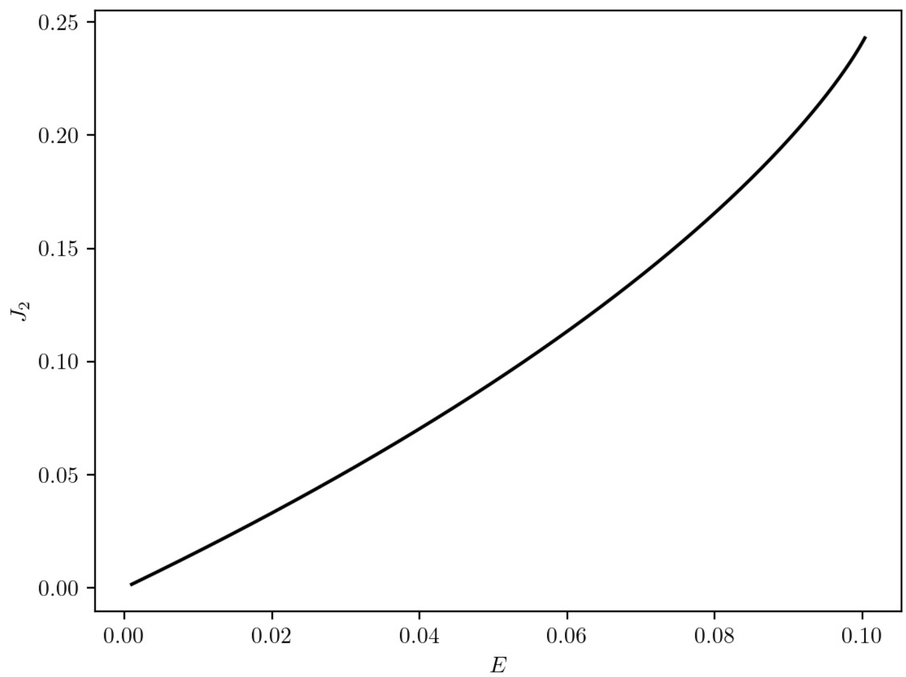

Figure 5.10: This graph shows a plot of (5.4.17) with the integral calculated numerically from $E = 0.001$ to $E = \ell_p^2/\pi^2 - 0.001$ with $k = 2$, $\ell_p = \ell_q = 1$.

The action variables for the interval $E \in (0, \frac{\ell_p^2}{\pi^2})$ have the following form

$$
J_1 = \frac{\kappa \ell_q \ell_p}{2\pi}; \qquad (5.4.18)
$$

$$
J_2 = \frac{\ell_p \ell_q}{\pi^2 k} \int_0^{\arccos\left(1 - \frac{2\pi^2 E}{\ell_p^2}\right)} \arccos\left(2 - \frac{2\pi^2 E}{\ell_p^2} - \cos\left(x^2\right)\right) \mathrm{d}q^2. \qquad (5.4.19)
$$

We therefore can find the Hamiltonian implicitly from the above to obtain $\overline{H}(J_2) = E$. The energy contour in the (J_1, J_2)-plane is therefore a straight line.

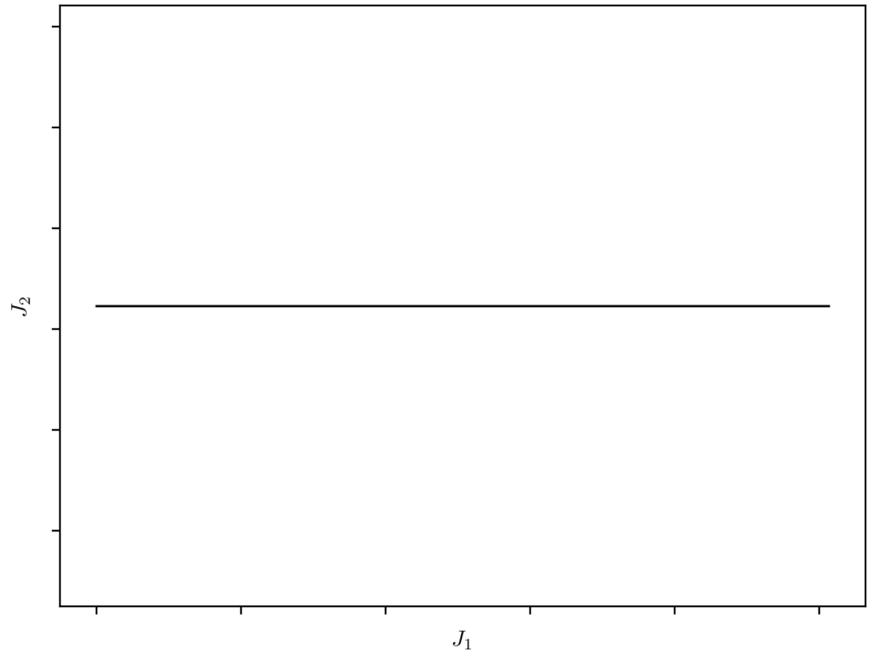

Figure 5.11: This graph shows a plot of the energy contour in the action space. Since the contour is a straight line the curvature is zero and therefore this system suffers from the same floor as the harmonic oscillator when computing the Berry-Tabor trace formula.

The frequencies are found by looking at the partial derivative of the Hamiltonian with respect to the action variable. Since we are unable to obtain an explicit formula for the Hamiltonian, we take the derivative of the actions with respect to the energy. We can then take the ratio of this to obtain the frequency. The the frequencies $\omega^i(\boldsymbol{J})$, are found to be

$$\omega^1(\boldsymbol{J}) = 0; \tag{5.4.20}$$

$$\omega^2(J_2) = \frac{\partial \overline{H}}{\partial J_2}. \tag{5.4.21}$$

In this case it is easier to use the property of one dimensional derivatives to find the frequency as a function of energy i.e.,

$$\omega^2(E) = \left(\frac{\partial J_2(E)}{\partial E}\right)^{-1} = \left(\frac{2\ell_q}{\ell_p k}\int_0^{\arccos\left(1-\frac{2\pi^2 E}{\ell_p^2}\right)} \frac{\mathrm{d}q^2}{\sqrt{1-(2-\frac{2\pi^2 E}{\ell_p^2}-\cos(q^2))^2}}\right)^{-1}, \tag{5.4.22}$$

With the period $t(E) = 2\pi/\omega^2$ (not squared), plotted below.

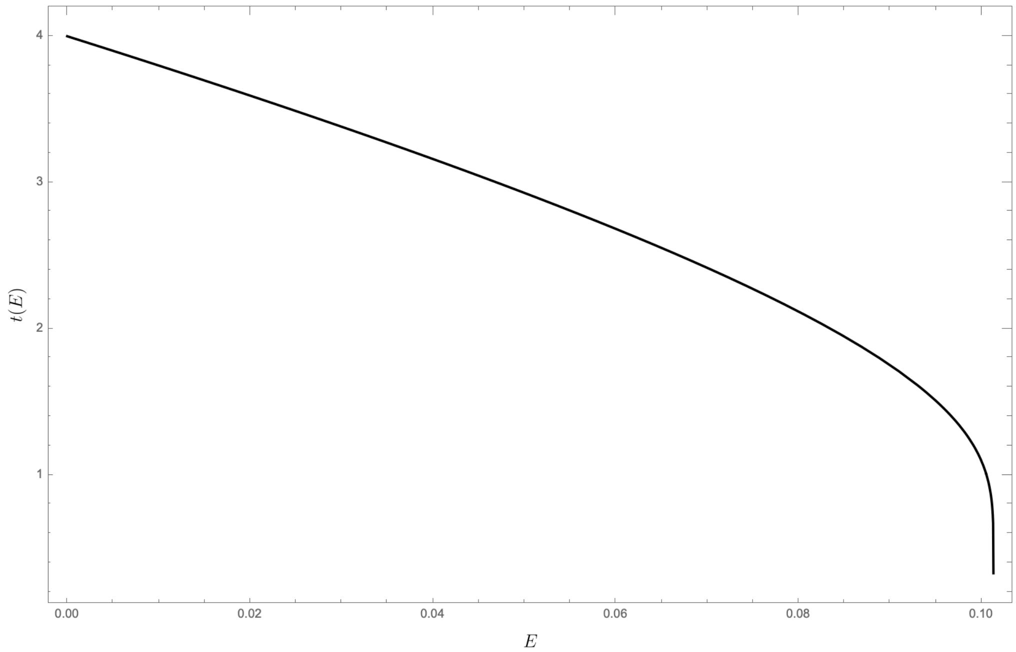

Figure 5.12: The period, $t(E) = 2\pi/\omega^2$ with respect to $E \in (0, \frac{\ell_p^2}{\pi^2})$ with the following parameter values: $\ell_p = 1$, $\ell_q = 1$, $k = 2$.

With the frequencies obtained we are thus in a position to obtain the equations of motion:

$$J_1(t) = C_1; \tag{5.4.23}$$

$$J_2(t) = C_2; \tag{5.4.24}$$

$$\theta^1(t) = \phi^1; \tag{5.4.25}$$

$$\theta^2(t) = \omega^2(J_2)t + \phi^2. \tag{5.4.26}$$

5.4.4 $E \in \left(\frac{\ell_p^2}{\pi^2}, \frac{2\ell_p^2}{\pi^2}\right)$

For the second energy interval we compute the action integrals in a similar fashion. The turning points of the new energy range are found by solving the following $p_2^0(q^m, \kappa) = \ell_p/2$. We see that changing κ, shifts the orbits, not changing the area enclosed as was the case for the first energy range. This again gives evidence that when calculating J_2, J_1 does not appear in the final form. This means that the Hamiltonian will contain one action variable only, thus making the energy contour have zero curvature.

With this we see that the turning points are thus found as

$$q^m = \pm\frac{\ell_q}{2\pi k}\arccos\left(3 - \frac{2\pi^2 E}{\ell_p^2}\right) - \frac{\ell_q \kappa}{k} + \frac{\ell_q m}{k}. \tag{5.4.27}$$

We take as our left turning point

$$\begin{aligned} a &= \frac{\ell_q}{2\pi k}\arccos\left(3 - \frac{2\pi^2 E}{\ell_p^2}\right) - \frac{\ell_q \kappa}{\ell_p k}, \\ &= \frac{\ell_q \beta}{2\pi k} - \frac{\ell_q \kappa}{\ell_p k} \end{aligned} \tag{5.4.28}$$

and as our right turning point

$$\begin{aligned} b &= \frac{\ell_q}{k} - \frac{\ell_q}{2\pi k}\arccos\left(3 - \frac{2\pi^2 E}{\ell_p^2}\right) - \frac{\ell_q \kappa}{\ell_p k}, \\ &= \frac{\ell_q}{k} - \frac{\ell_q \beta}{2\pi k} - \frac{\ell_q \kappa}{\ell_p k}, \end{aligned} \tag{5.4.29}$$

where we have set $\beta = \arccos\left(3 - \frac{2\pi^2 E}{\ell_p^2}\right)$. J_1 is calculated the same as for the first energy interval. The calculation of J_2 is similar to that of the first energy interval. Taking into account the anticlockwise sense of the orbit yields

$$\begin{aligned} J_2 &= \int_a^b (\ell_p + p_2) - (-p_2)\,\mathrm{d}q^2 \\ &= \ell_p(b - a) + 2\int_a^b p_2\,\mathrm{d}q^2. \end{aligned}$$

Simplifying the integral we obtain

$$J_2 = \frac{\ell_p \ell_q}{k} - \frac{\ell_p \ell_q}{\pi k} \arccos\left(3 - \frac{2\pi^2 E}{\ell_p^2}\right)$$
$$+ \frac{\ell_q \ell_p}{2\pi^2 k} \int_{\arccos\left(3-\frac{2\pi^2 E}{\ell_p^2}\right)}^{2\pi - \arccos\left(3-\frac{2\pi^2 E}{\ell_p^2}\right)} \arccos\left(2 - \frac{2\pi^2 E}{\ell_p^2} - \cos\left(q^2\right)\right) \, \mathrm{d}q^2. \quad (5.4.30)$$

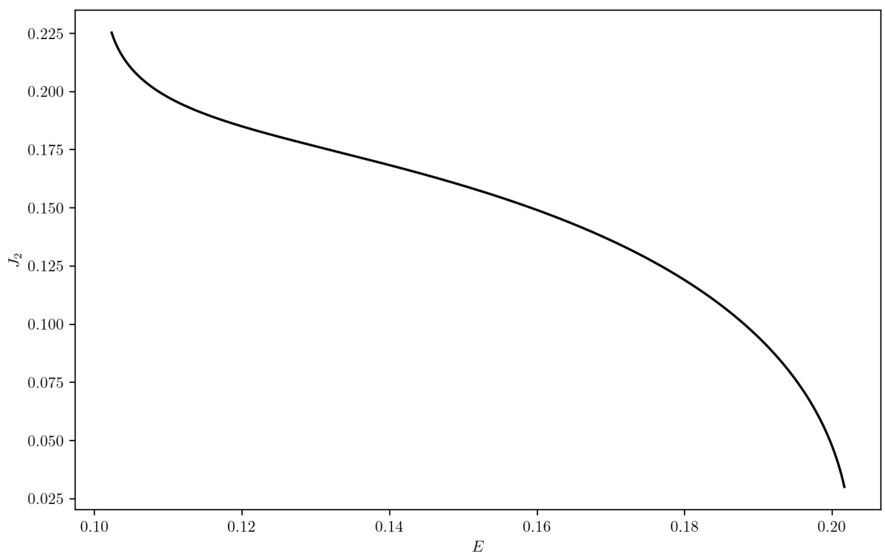

Figure 5.13: This graph shows a plot of (5.4.30) with the integral calculated numerically from $E = 0.001 + \ell_p^2/\pi^2$ to $E = 2\ell_p^2/\pi^2 - 0.001$ with $k = 2$, $\ell_p = \ell_q = 1$.

Where again taking the reciprocal

$$\omega^2(E) = \left(\frac{\partial J_2}{\partial E}\right)^{-1} = \left(-\frac{2\pi \ell_q}{k\ell_p \sqrt{\frac{3\pi^2 E}{\ell_p^2} - 2 - \frac{E^2\pi^4}{\ell_p^4}}} \quad (5.4.31)\right.$$
$$\left. + \frac{\ell_q}{\ell_p k} \int_{\arccos\left(3-\frac{2\pi^2 E}{\ell_p^2}\right)}^{2\pi - \arccos\left(3-\frac{2\pi^2 E}{\ell_p^2}\right)} \frac{\mathrm{d}q^2}{\sqrt{1 - \left(2 - \frac{2\pi^2 E}{\ell_p^2} - \cos(q^2)\right)^2}}\right)^{-1}. \quad (5.4.32)$$

We note that the first term on the right-hand side dominates and thus the frequency is negative. From figure 5.9, the orbits in the energy interval

$(\ell_p^2/\pi^2, 2\ell_p^2/\pi^2)$ are anticlockwise in orientation, thus the frequency has an opposite sign. The plot below is of the period $t(E) = -2\pi/\omega^2$ (not squared) where the minus sign is included to ignore the orientation of the orbit. Note that close to the separatrix the period tends to infinity, which is consistent with the definition of the separatrix.

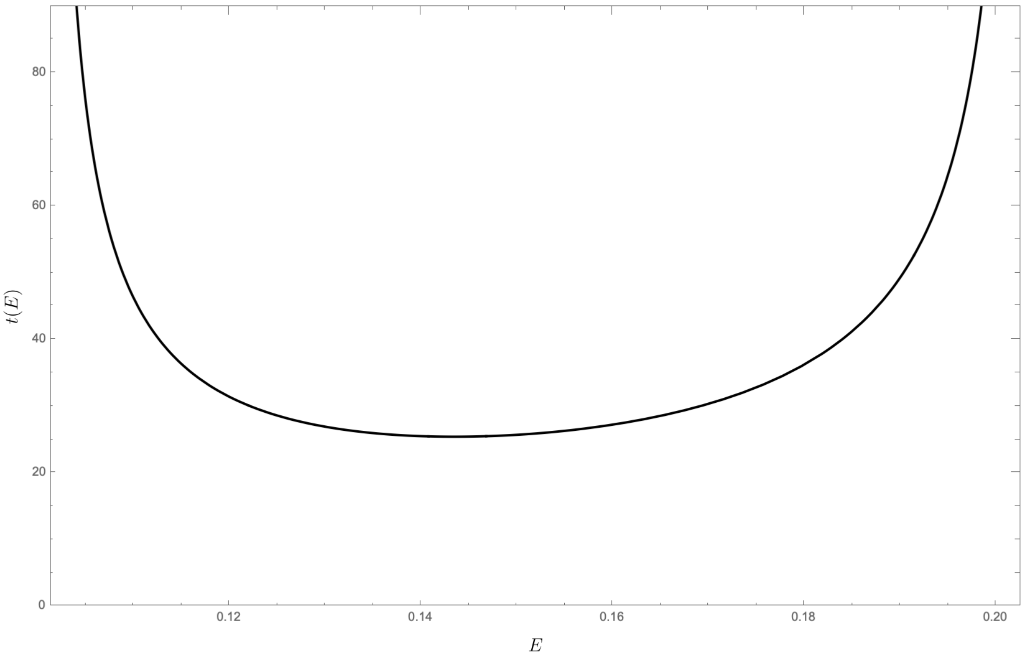

Figure 5.14: The period, $t(E) = -1/\omega^2(E)$ with respect to $E \in (\frac{\ell_p^2}{\pi^2}, \frac{2\ell_p^2}{\pi^2})$ with the following parameter values: $\ell_p = 1$, $\ell_q = 1$, $k = 2$.

The equations of motion are found similarly to the previous case

$$J_1(t) = c_1 \tag{5.4.33}$$

$$J_2(t) = c_2 \tag{5.4.34}$$

$$\phi^1(t) = \phi^1 \tag{5.4.35}$$

$$\phi^2(t) = \omega^2(J_2)t + \phi^2. \tag{5.4.36}$$

Keeping E fixed the energy contour in action space is thus parameterised by κ, which is equivalent to the ξ coordinate in [MVB76]. As we have shown, the energy contour is a straight horizontal line as shown in figure 5.11. The curvature of the energy contour therefore vanishes and the trace formula in [MVB76] cannot be used, note that in [MVB77a] it was shown that the two methods for computing the trace formula are equivalent. One thus turns to the uniform approximation also described in [MVB76], to account for the zero curvature of the energy contour. This uniform approximation was first implemented to take into account of negative values of the action, which give rise to complex orbit contributions to the trace formula. The modified trace formula is an integration over the energy contour, and since we are in two dimensions, the energy contour is parameterised by one variable. The integration limits correspond to where the system exhibits one dimensionally periodic motion. The modified trace formula is, where we have labelled the parameterisation of the energy contour as η,

$$d(E) = n_W(E) + \frac{1}{\hbar^2}\sum_{\boldsymbol{M}}{}' \int_{\eta_1}^{\eta_2} \frac{\mathrm{e}^{\frac{2\pi i}{\hbar}\langle \boldsymbol{M}, \boldsymbol{J}(\eta)\rangle}}{|\boldsymbol{\omega}(\boldsymbol{J}(\eta))|}\mathrm{d}\eta \tag{5.4.37}$$

The uniform approximation is derived by using an integral with the same qualitative behaviour as (5.4.37). Due to the form of the action variables, we do not need this uniform approximation explicitly. Since the energy contour is parameterised by κ we can use this as the integration variable in (5.4.37). Since, J_2 is independent of κ it is seen as a constant. The phase can thus be written as

$$\frac{2\pi\langle \boldsymbol{M}, \boldsymbol{J}(\kappa)\rangle}{\hbar} = A + B\kappa, \tag{5.4.38}$$

where

$$A = \frac{2\pi M_2 J_2(E)}{\hbar}, \text{ and } B = \frac{\ell_q \ell_p M_1}{\hbar}. \tag{5.4.39}$$

Substituting this into (5.4.37) and using $\eta_2 = 0$, and $\eta_1 = 1$, we find

$$\begin{aligned} d(E) &= \frac{1}{\hbar^2}\sum_{\boldsymbol{M}}{}' \int_1^0 \frac{\mathrm{e}^{Ai+Bi\kappa}}{|\omega_0|}\mathrm{d}\kappa, \\ &= -\frac{2}{\hbar^2}\sum_{\boldsymbol{M}}{}' \mathrm{e}^{\frac{2\pi i M_2 J_2(E)}{\hbar}+\frac{\ell_q\ell_p i M_1}{2\hbar}} \sin\left(\frac{\ell_q\ell_p M_1}{2\hbar}\right) \end{aligned} \tag{5.4.40}$$

where since the Hamiltonian is a function of J_2 only, $\omega^1 = 0$ and thus only ω^2 contributes and is equal to a constant ω_0. Using the phase space quantisation condition $2\pi\hbar N = \ell_q \ell_p$, we find (5.4.40) can be written as

$$\begin{aligned} d(E) &= -\frac{2}{\hbar^2} \sum_{\boldsymbol{M}}{}' \mathrm{e}^{\frac{2\pi i M_2 J_2(E)}{\hbar} + \pi i N M_1} \sin\left(\pi N M_1\right) \\ &= 0, \end{aligned} \tag{5.4.41}$$

since $NM_1 \in \mathbb{N}$.

The uniform approximation used here thus results in zero for the density of states. It is thus clear that the for the Landau Hamiltonian the Berry-Tabor trace formula is unable to provide higher order corrections to the density of states.

Chapter 6

Summary and outlook

The main goal of this thesis was to analyse the spectra of operators obtained via a magnetic Weyl calculus on a torus phase space and obtain a semiclassical trace formula for the density of states. We have found that we were successfully able to quantise a periodic symbol on the torus in the presence of a magnetic field. From the compact nature of the phase space quantisation conditions were obtained and a general phase space quantisation condition regardless of any symbol and a flux quantisation condition for homogeneous and non-homogeneous magnetic fields.

We successfully developed an algorithm to obtain the Hamiltonian matrix for an arbitrary vector potential, and compared the results with that of the non-magnetic Laplacian with which the results agreed to an error of 10^{-16}. This algorithm can be used to populate any matrix in which periodic boundary conditions are used. This is proves useful in the analysis of discretise partial differential equations.

We have also proved that for the case of the torus phase space, the minimal coupling can only be used if the vector potential is linear and $M/N = k \in \mathbb{Z}$. With these two conditions we were able to write down the classical Hamiltonian for the Landau gauge. We then proposed a non-homogeneous magnetic field, based loosely on a Fourier expansion of the Henon-Heiles potential, a known potential for an non-integrable system. We analysed the spectra numerically and computed well known spectral statistics. We showed that due to degeneracy, the eigenvalues statistics were inconclusive; increasing the non-homogeneity broke the degeneracy and spectral statistics were used to analysis the nature of these systems. We obtained the result whereby increasing the non-homogeneity transforms the system into a non-integrable system with a finite energy interval.

Using the Berry Tabor formulation, we obtained a trace formula for the non-magnetic case which agrees well with the computed spectrum as well as the smoothed density of states. The formulation of Berry-Tabor was however unsuccessful in obtaining the trace formula for the case of the Landau gauge. This arose because of the failure of the Berry-Tabor uniform approximation which has been shown to yield successful trace formulae for systems with an energy surface of zero curvature.

For future work the trace formula for the case of non-homogeneous magnetic field could be attempted. Perhaps other methods could be used to obtain the classical Hamiltonian in this case, for example, a suitable approximation scheme which approximates discrete derivatives. Once a Hamiltonian is obtained, the trace formula could be tackled, quite possibly the Gutzwiller trace formula for non-integrable systems.

route would be to find a suitable uniform approximation which corrects the Berry-Tabor trace formula for the Landau gauge. A deeper look into the Laplacian could also prove useful, since for the classical system a simple linear canonical transformation yields a Hamiltonian which does not depend on (q^1, p_1), however this changes the fundament domain to a parallelepiped and the Weyl calculus on the torus is no-longer applicable. This suggests generalising the Weyl calculus to general domains in order to treat a wider class of systems.

The theory of Landau levels plays a major role in the study of the quantum Hall effect, the role of a torus configuration space has been extensively used [Fre13, MGT16, Fre15, Koh85], but the case of the torus phase space has not been studied. It would be interesting to study the effect that phase space quantisation condition would have in relation to the quantum Hall effect.

In this thesis, we looked at the case where the vector potential was not quantised. The most accurate theory of electromagnetism at the quantum scale is quantum electrodynamics (QED), here the vector potential is quantised and expanded in a basis of creation-annihilation operators. One could expand Weyl quantisation on the torus to incorporate QED into the formulation.

Another possible line of future work could be the generalisation of magnetic Weyl quantisation to include other gauge potentials including non-abelian potentials and investigate the consequences this has on the Weyl calculus on the torus.

Appendices

Appendix A

Method of Stationary Phase

The method of stationary phase is used in semiclassical physics when studying the asymptotics in the limit $\hbar \to 0$ of integrals of the form

$$I_\hbar(a, \phi(\boldsymbol{x})) = \int_{\mathbb{R}^n} \mathrm{e}^{i\frac{\phi(\boldsymbol{x})}{\hbar}} a(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \tag{A.0.1}$$

where $a(\boldsymbol{x}) \in C_c^\infty(\mathbb{R}^n)$ and $\phi(\boldsymbol{x}) \in C^\infty(\mathbb{R}^n)$. As $\hbar \to 0$, the exponent becomes rapidly oscillating and the integral picks up positive and negative contributions causing cancellations for nearby points.

Lemma A.0.1. *If $\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}) \neq 0$ on the support of $a(\boldsymbol{x})$ then $I_\hbar = \mathcal{O}(\hbar^\infty)$.*

Proof. We define an operator

$$L := \frac{\hbar}{i} \frac{1}{|\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x})|^2} \langle \nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}), \nabla_{\boldsymbol{x}} \rangle$$

for all $\boldsymbol{x}$ in the support of $a(\boldsymbol{x})$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. We notice that $L\left(\mathrm{e}^{i\frac{\phi(\boldsymbol{x})}{\hbar}}\right) = \mathrm{e}^{i\frac{\phi(\boldsymbol{x})}{\hbar}}$ and therefore applying it N times we find

$$I_\hbar(a, \phi(\boldsymbol{x})) = \int_{\mathbb{R}^n} L^N \left(\mathrm{e}^{i\frac{\phi(\boldsymbol{x})}{\hbar}}\right) a(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

Using integration by parts and noting that boundary terms are zero since the limits are outside the support, we find

$$I_\hbar(a, \phi(\boldsymbol{x})) = \int_{\mathbb{R}^n} \mathrm{e}^{i\frac{\phi(\boldsymbol{x})}{\hbar}} (L^*)^N (a(\boldsymbol{x})) \, \mathrm{d}\boldsymbol{x}.$$

Taking the modulus and noting that the operator is $\mathcal{O}(\hbar)$ we obtain the result. $\square$

The formula for the method of stationary phase can be put in the form of the following theorem (for a detailed discussion of the method of stationary phase, see [Zwo12], [AGH94] and [Hör90]).

Theorem A.0.2. *Let $a(\boldsymbol{x}) \in C_c^\infty(\mathbb{R}^n)$ and $\boldsymbol{x}_0$ is a point in the support of $a(\boldsymbol{x})$, with $\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}_0) = 0$, $\frac{\partial^2\phi(\boldsymbol{x}_0)}{\partial x_i \partial x_j} \neq 0$ and $a(\boldsymbol{x}_0) \neq 0$ where Hessian matrix evaluated at $\boldsymbol{x}_0$ is defined to be $\frac{\partial^2\phi(\boldsymbol{x}_0)}{\partial x_i \partial x_j}$, then*

$$I_\hbar = (2\pi\hbar)^{\frac{n}{2}} \left|\det \left(\frac{\partial^2\phi(\boldsymbol{x}_0)}{\partial x_i \partial x_j}\right)\right|^{-\frac{1}{2}} a(\boldsymbol{x}_0)\mathrm{e}^{\frac{i}{\hbar}\phi(\boldsymbol{x}_0) + \frac{\pi i}{4}\beta} + \mathcal{O}(\hbar^{\frac{n+2}{2}}), \quad \text{(A.0.2)}$$

where β is the sign of the Hessian, which is the difference in the number of positive eigenvalues and negative eigenvalues.

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